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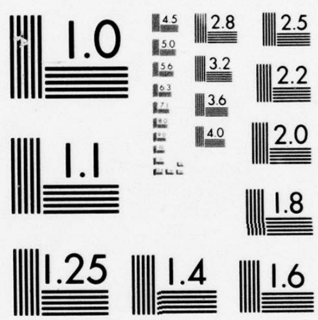

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by

EDUARDO M. MODIANO

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FOREWORD

The Operations Research Center at the Massachusetts Institute of Technology is an interdepartmental activity devoted to graduate education and research in the field of operations research. The work of the Center is supported by government grants and contracts. The work reported, herein, was supported, in part, by the U.S. Army Research Office under Contract DAAG29-76-C-0064.

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ABSTRACT

The major theme of this work is the integration of depletable resources supply and sectoral dynamic models. These models are inherently normative due to their mathematical programming formulations. The economic sectors that demand the depletable resource minimize in each time period the cost of meeting the demand for end-use goods. The suppliers allocate the fixed resource stock over time such as to maximize the net present value of profits. Equilibrium is defined by linking the suppliers' revenue with sectoral savings in costs. Mathematical programming concepts, particularly decomposition methods, facilitate the integration of these models. Some extensions are discussed and implemented in this context: essential versus inessential resources, the existence of institutional agents and capacity constraints on resource extraction.

The existence of equilibrium in the case of multiple collusive and competing economic agents is also considered. Competing multiple depletable resource suppliers create several difficulties in our basic framework. These are also mentioned. Models are developed to consider the major source of uncertainties in depletable resource intertemporal planning: technological transition, end-use demands and resource reserves. The techniques used are dynamic programming, linear programming with recourse and Markovian decision theory, respectively.

Finite-horizon approximation methods for the case of infinite planning horizons are presented. These are of particular interest in the integration of transient and stationary stage models. Leontief substitution systems are also considered. Finite-elasticities in the demand for end-use goods and the supply of alternative primary supplies add more flexibility to the equilibrium model. Linear relationships can be effectively handled by the development of parametric quadratic programming.

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Chapter 1: Preface

1. I. Planning with Depletable Resources

Depletable resources are a special type of capital assets because the existing stocks are not reproducible or augmentable. There are a number of natural resources that at the present stage of technology fall into this category. Examples are oil, coal, copper, uranium, lead and zinc, to cite only a few. However, technological progress might lead to methods of recycling these resources and thereby make them effectively inexhaustible. Then the depletable resources need no longer be treated differently from ordinary capital assets.

The depletable resource planning problem is essentially an intertemporal problem. The trade-off that exists in resource depletion is basically one of "now versus future". The more (less) the resource is used up now, the less (more) will be available in the future. In the case of private ownership the supplier(s) of a depletable resource face the decision of whether or not to hold back stocks from the market in the expectation of higher future prices. Whether the resource is publicly or privately owned current decisions about natural resource depletion affect the welfare of future generations. If a socially optimal depletion pattern (see section 1.II.) exists it can be used as a benchmark for evaluating current strategies and the implications of market imperfections. Any conjectures about optimality in natural resource depletion either at the supplier's level or from a social standpoint requires knowledge or assumptions about the future.

Planning exhaustion at the supplier level requires knowledge of the time path of prices and/or demand as well as of the technological progress in resource extraction. This is further complicated by the lack of forward markets and risk markets for most types of depletable resources. Sellers will have to base their decisions on individual expectations about future demand. The formation of these expectations are likely to be influenced by individual expectations about technological progress in resource usage, the rate of new discoveries and the size of total reserves. The complex implications of individual expectations for natural resource exhaustion have been explored by Heal [38].

Besides technological progress that may render an extremely valuable resource worthless to the resource owners sometime in the future, the other source of uncertainty faced in resource exhaustion planning is the actual size of reserves. The size of the stock of an exhaustible resource is fixed but unknown†. Most empirical work (e.g. Nordhaus[66]) has been in general based on what is called "proved reserves", which accounts only for a small fraction of the existing stock. Most "doomsday" predictions have been based on a comparison of current levels of consumption with this measure of resource availability. Solow [82] describes "proved reserves" as an economical concept rather than geological and defines it as

> "... the volume of reserves that interested parties have found it profitable to find, map and measure in the light of their own planned requirements ...."

---

†Kaufman [49] showed that the distribution of sizes of oil (gas) fields as deposed by nature is, to a first approximation, lognormal.

Therefore it seems that "proved reserves" are not a reliable measure of resource availability.†

Planning at the national level requires the consideration of the intertemporal biases in resource extraction that are likely to result from market imperfections characteristic of natural resources markets. Monopolistic practices that seem to plague the market for natural resources have been shown by Weinstein and Zeckhauser [90,91] to result in underexploitation in most situations. Sweeney [87] addresses the intertemporal effects in exhaustion patterns resulting from different market forces and some forms of government regulations.

A related question in depletable resource planning is the determination of prices. Compared to reproducible assets the prices of resources whose stock if fixed should include a scarcity component (royalty or rent). Prices are likely to show a permanent movement in time to reflect its increasing scarcity if technological progress in resource usage is not strong enough to compensate for the force of depletion. Hotelling's [44] work was the first formal attempt to prove this assertion (see section 1.II.1.). In an efficient market we should expect current prices to be influenced by the uncertainties described above, future expectations, and by the time path of prices of actual and/or potential substitutes as well. Also, market structure and market forces will be reflected in observable current prices and to some extent will tend to bias the

†The disparities have been illustrated by Solow [82]. Taking the example of lead, the U.S. Geological Survey shows that at current consumption rates proved reserves of lead account for 10 years left, potential reserves for 162 years left, and crustal abundance for 85 x 10^6 years left.

price of the resources away from its social value.

This short exposition of problems of depletable resource exhaustion makes it clear why considerable attention to modelling depletable resource allocation is being devoted by economists, environmentalists, engineers and operations researchers. The formation of the OPEC cartel and the proposition of Meadows et al. [63] that present "inappropriate" depletion patterns leading to ultimate exhaustion of natural resources would have catastrophic consequences for society have led to a revival of research effort in resource depletion planning.

Several mathematical models for optimal resource allocation have been devised to prove and disprove assertions about natural resource depletion. In the next section we will review most categories of models and illustrate, where possible, the conclusions that have been drawn from their application. While the list of contributions described below does not intend to be exhaustive, it illustrates the different approaches utilized so far to model depletable resources exhaustion. We shall try to keep notation as much as possible compatible between models and with this thesis. The third section of this chapter gives an overview of the specific contributions of this thesis.

1. II. Modelling Efforts in Resource Depletion

We shall review here a sample of models of depletable resources. For this purpose these models have been divided into five categories:
1) Supply Models, 2) Optimal Economic Growth Models, 3) Sectoral Models,
4) Equilibrium Models, and 5) Econometric Models.

1. II. 1. Supply Models

Supply models are concerned with the behavior of individual firms facing known prices or demand functions over the planning horizon. Included in this category are the first attempts to formalize what is referred to in the literature as the "Economic Theory of Exhaustible Resources".

In what follows we let r_t denote the amount extracted at time t, S_t denote the stock of the resource at the time t and δ the continuous-time discount rate. Hotelling's [44] economic theory of exhaustible resources is the classic work in this category. The most general problem faced by a supplier of a depletable resource studied by Hotelling was to†

$$\text{maximize} \int_0^\infty e^{-\delta t} p_t(S_0 - S_t, r_t) r_t dt$$

$$\text{subject to} \quad \dot{S}_t = -r_t \tag{1.1a}$$

$$\int_0^\infty r_t d_t \le S_0 \tag{1.1b}$$

$$r_t \ge 0 \tag{1.1c}$$

In this formulation p_t is the net price received after deducting the costs of extraction at time t. Net prices, (p_t) are allowed to depend not only on the current rate of extraction (r_t) but also on accumulated extraction (S_0-S_t). The criterion to be maximized is the net present value of receipts. The case of perfect competition where each individual supplier is a price-taker ($p_t(S_0-S_t,r_t) = p_t(S_0-S_t)$) is just a special case. In the latter, when dependency on accumulated extraction is dropped

†The dot will denote the time derivative.

$(p_t(S_0-S_t, r_t) = p_t))$, the maximization of the social value of the resource will lead to a time path of prices increasing at the discount rate $(p_t = p_0 e^{\delta t})$.

Gordon [28] reformulated the above problem introducing explicitly extraction cost curves and proposed to

$$\text{Maximize} \int_0^\infty e^{-\delta t}\, \Pi_t(S_0 - S_t, r_t)dt$$

$$\text{subject to (1.1a), (1.1b) and (1.1c),}$$

where profits at time t, Π_t, are allowed to depend on both current (r_t) and cumulative extraction (S_0-S_t). When dependence on cumulative extraction is dropped the optimality conditions imply that for positive extraction rates $(r_t>0)$,

$$\frac{\partial \Pi_t}{\partial r_t} = \delta e^{\delta t} \tag{1.2}$$

will hold. The right hand side of (1.2) has been interpreted by Scott [75] as a user's cost, a measure of the sacrifice of future revenue due to present sales. Therefore (1.2) can be read as a requirement that marginal total profits including the imputed user's cost be zero. In the special case of perfect competition Gordon proved Herfindahl's [41] assertion that Hotelling's results are only compatible with a constant cost industry. The profit per unit (p_t) will only be constant when the cost function is a linear function of the amount extracted r_t; namely, when marginal costs equal average costs at all extraction levels.

Cummings [10] explored two situations. Case I is that of N indepen-

dent firms sharing a common resource stock and a central authority acting to determine each individual firm's resource use r_t^i at time t so as to maximize the joint present value of profits for the resource owners as a whole. The problem faced by the central authority is then to

$$\text{Maximize} \int_0^\infty e^{-\delta t} \sum_{i=1}^N \Pi_t^i (S_0 - S_t, r_t^i) dt$$

$$\text{subject to } \dot{S}_t = - \sum_{i=1}^N r_t^i$$

$$\int_0^\infty \sum_{i=1}^N r_t^i dt \leq S_0$$

$$r_t^i \leq \bar{r}_t^i (S_0 - S_t) \qquad i = 1,2,\ldots,N$$

$$r_t^i \geq 0 \qquad i = 1,2,\ldots,N.$$

This formulation introduces upper bounds ($\bar{r}_t^i$, i=1,2,...,N) on the amount that may be extracted by each individual firm at time t. The upper bounding is functionally dependent upon cumulative extraction at time t. The optimality conditions for the problem above in the case the extraction rates for all firms are positive and do not attain their upper bounds ($0 < r_t^i < \bar{r}_t^i(S_0 - S_t)$, i=1,2,...,N) imply that the central authority will act such as to equate the marginal costs of individual firms plus a measure of user's cost with marginal joint revenue.

Case II is one in which each individual owns a fraction of the resource stock and acts individually. Making N=1 in the above formulation, this model differs from Gordon's model essentially by the inclusion of

upper bounds on extraction rates.

Weinstein and Zeckhauser [91] re-derived some of the above results for the special cases of competitive and monopolistic behavior with zero, constant and increasing marginal costs. They showed also that in a competitive market with zero extraction costs and random prices, risk neutral suppliers would choose an allocation such that expected prices grow at the interest rate. This results from the maximization of expected discounted consumers' surplus.

1. II. 2. Optimal Economic Growth Models

The recognition that some resources that enter into the economy's production process are limited in supply has led to a revision of economic growth theory. Optimal growth models could not disregard the resource constraint imposed on the economy's growth potential by the fixity of the resource stock. Some of the questions that arise when studying exhaustion of a depletable resource, such as optimal extraction rate, efficiency prices, time of exhaustion, have been imbedded in the more general question of optimal economic growth.

The characteristics of a depletable resource that plays a crucial role in determining an optimal path to exhaustion is its essentiality to the economy, essentiality being defined as the output being nil in the absence of the depletable resource. In the case of an essential resource one might want to know whether to deplete the resource gradually over the planning horizon or exhaust it in finite time and live off the capital stock existing by the time of depletion. When the resource is inessential society may rely on reproducible inputs to continue production after the

resource is depleted in finite time. In both situations the ease with which the reproducible inputs can be substituted for the depletable resource will be an important factor determining the optimal path to exhaustion.

There has been significant literature examining the properties of optimal plans derived from resource constrained economic growth models. The different models proposed differ on their assumptions about essentiality, planning horizon and social preferences, to cite a few. The contribution of different authors will be discussed in the framework of a more general model formulated below.

We suppose the existence of a depletable resource that may be either consumed or used in production and of a composite consumption good (capital) that combined with the depletable resource and labor can reproduce itself. Consumption at time t is given by the pair (C_t, r_t^c) where C_t denotes consumption of the composite good and r_t^c denotes consumption of the depletable resource at time t. A social intertemporal preference structure $\mathcal{U}(C, r^c)$ specified exogenously provides a numerical evaluation of the entire consumption program.

Let K_t, L_t and r_t^p stand respectively for the stock of capital, the labor force size and the amount of depletable resource used in production at time t. Efficient output possibilities are given by an instantaneous production function $F^t(K_t, L_t, r_t^p)$. The initial stocks of capital, K_0, and of the depletable resource, S_0, are given. The labor force dynamics are specified exogenously by $L(t, L_0)$ where L_0 is the starting labor force size. We let Ω denote a set of constraints imposed on depletable resource

consumption during the planning horizon. Furthermore, we allow capital depreciation at a rate ν.

Over the planning horizon T (possibly infinite) the problem then becomes

$$
\begin{array}{llll}
\text{maximize } \mathcal{U}(C,r^c) & & & \underline{\text{shadow prices}} \\
\text{subject to } \dot{K}_t + \nu K_t = F^t(K_t,L_t,r_t^p) - C_t & & 0 \le t \le T & p_t \\
\quad \int_0^T (r_t^c + r_t^p)dt \le S_0 & & & \lambda \\
\quad r_t^c \in \Omega & & 0 \le t \le T & \\
\quad K_t, r_t^c, r_t^p, C_t \ge 0 & & 0 \le t \le T &
\end{array}
$$

with K_0, S_0 and L_0 given and $L_t = L(t,L_0)$. When the planning horizon T is finite, a set of terminal conditions on the capital and resource stock may be specified.

Table 1.1 relates the above formulation with some of the important contributions to the field of resource-constrained optimal economic growth. Anderson's [1] criterion is to maximize the discounted sum of per capita consumption of the composite consumption good over a finite planning horizon to determine the optimum savings rate path s defined as

$$s = 1 - \frac{C_t}{F(K_t,L_t)} .$$

He shows that in the optimal path the undiscounted shadow price of the depletable resource grows exponentially at a rate $(\delta + \pi_p - \pi_1)$ which can be regarded as the social discount rate adjusted for the growth of the

Table 1.1 Optimal Economic Growth Models

Conditions:	Anderson [1]	DasGupta & Heal [14]	Koopmans [52] Model I	Koopmans [52] Model II	Solow [81]	Stiglitz [84]
Decision set Ω	{0}	{0}	$\{r^c \geq \underline{r}^c\}$: $\underline{r}^c = 0$	$\{r^c \geq \underline{r}^c\}$: $\underline{r}^c > 0$	{0}	{0}
Planning horizon T	finite	∞	∞	finite	∞	∞
Decision variables	$\left\{C_t, K_t, r_t^P\right\}$	$\left\{C_t, K_t, r_t^P\right\}$	$\left\{C_t, K_t, r_t^c\right\}$	$\left\{C_t, K_t, r_t^c, T\right\}$	$\left\{C_t, K_t, r_t^P\right\}$	$\left\{C_t, K_t, r_t^P\right\}$
Intertemporal preferences $U(C, r^c)$	$\int e^{-\delta t}(C_t/P_t)dt$	$\int e^{-\delta t}u(C_t)dt$	$\int e^{-\delta t}[u(C_t) + v(r_t^c)]dt$	$\int e^{-\delta t}[u(C_t) + v(r_t^c)]dt$	$\inf_t C_t/L_t$	$\int L_t u(C_t/L_t)e^{-(\delta-\pi_\ell)t}dt$
Labor Force Size L_t	$L_0 e^{\pi_\ell t}$	L_0	L_0	L_0	$L_0 e^{\pi_\ell t}$	$L_0 e^{\pi_\ell t}$
Production function $F^t(K_t, L_t, r_t^P)$	$\min\left\{F(K_t, L_t), e^{\pi_r t} r_t^P\right\}$	$F(K_t, r_t^P)$	$F(K_t)$	$F(K_t)$	$(L_t e^{\pi_h t})^g (r_t^P)^h K_t^{1-g-h}$; I: $\pi_\ell=0$, $\pi_h=0$; II: $\pi_\ell=0$, $\pi_h>0$; III: $\pi_\ell>0$, $\pi_h=0$; IV: $\pi_\ell>0$, $\pi_h>0$	$K^{\alpha_1}L^{\alpha_2}r^{\alpha_3}e^{\lambda t}$; $\alpha_1+\alpha_2+\alpha_3 = 1$
Rate of capital decay ν	nonzero	0	0	0	I: 0; II: 0; III: 0; IV: 0	0
Additional constraints	Terminal conditions	none	none	$0 \leq T \leq \bar{T}$	none	none

Definitions

- δ = social rate of continuous time discounting
- P_t = population size at time t $\left(\equiv P_0 e^{\pi_p t}\right)$
- π_p = relative rate of population growth
- $F(\ldots)$ = neoclassical production function
- π_ℓ = relative rate of labor force growth
- $u(C_t)$ = instantaneous utility of consuming C_t
- $v(r^c)$ = instantaneous utility of consuming r_t^c
- $\underline{r}^c$ = subsistence level of resource consumption
- $\bar{T}$ = maximum survival period ($\equiv S_0/\underline{r}^c$)
- π_h = rate of Hicks-neutral technological progress
- λ = rate of technological progress
- π_r = rate of technological progress in resource requirement

idle population. The resultant optimum savings rate path, $\hat{s}$, can be divided into three stages. During the initial and last stages, the optimum savings rate is equal to unity. This peculiarity is due to the linearity of the utility function represented by the consumption per capita. The intermediate phase consists of what he calls a "resource-limited turnpike phase". Compared to the unconstrained optimal path, the author concludes that the resource-constrained model shows a tendency to postpone capital accumulation and to spend time in a turnpike growth path where capital is used less intensively. The problem of finite time depletion versus gradual depletion over the planning horizon has been neglected by Anderson [1] because of an imposed feasibility condition that precludes depletion during the finite planning horizon (T) considered.

DasGupta and Heal [14] in their work derived conditions determining whether or not the resource is exhausted in finite time. They have shown that the finite time depletion will occur if the marginal productivity of the resource approaches a finite limit as resource usage approaches zero and the output is positive in the absence of the resource as long as the stock of capital is non-zero. Namely, it is required for finite time depletion that

$$\lim_{r^P \to 0} \frac{\partial F(K,r^P)}{\partial r^P} < +\infty$$

and

$$F(K,0) > 0 \quad K > 0.$$

Finite time depletion rests not only on the resource being inessential (output being positive in the absence of the depletable resource) but also

on the behavior of its marginal product. From the optimality conditions, DasGupta and Heal have shown that the future value of the shadow price of the exhaustible resource relative to the efficiency price of the composite consumption good will be equal to the marginal product of the resource.

$$\frac{e^{\delta t}\lambda}{p_t} = \frac{\partial F\left(K_t, r_t^p\right)}{\partial r_t^p}$$

at positive levels of resource utilization ($r_t^p > 0$). Also equality between the rates of return on the two assets will result

$$\frac{\partial^2 F\left(K_t, r_t^p\right)}{\partial t \partial r_t^p} \cdot \frac{1}{\dfrac{\partial F\left(K_t, r_t^p\right)}{\partial r_t^p}} = \frac{\partial F\left(K_t, r_t^p\right)}{\partial K_t} .$$

The class of constant-elasticity production functions illustrates the major role played by the elasticity of substitution (σ) between K and r^p in determining the characteristics of the optimal plan. When $0 \leq \sigma \leq 1$, the resource is essential in the sense of output being nil in its absence. Otherwise it is inessential. The resource will be depleted in finite time only if $\sigma > 1$, and the future value of the shadow price of the resource in terms of the composite good will tend to infinity except in the Cobb-Douglas case ($\sigma = 1$).

Koopmans' [52] models differ from the previous ones because the depletable resource is not an input to production but only a consumption good. In this context the concept of essentiality refers to essentiality of the resource to life. If the resource is inessential to life then the

subsistence level of resource consumption is nil ($\underline{r}^c = 0$). Models I and II distinguish the case of inessentiality ($\underline{r}^c = 0$) and essentiality ($\underline{r}^c > 0$) of the depletable resource respectively. Model I breaks into a Ramseyan capital model and a discounted version of Gale's [25] "cake-eating" model. The optimal resource consumption path $\hat{r}^c_t$ is such that the present value of marginal utility is constant along the interval of time of positive consumption levels ($\hat{r}^c_t > 0$). The optimal path for the consumption of the composite good $\hat{C}_t$ is independent of the time path of consumption of the depletable resource $\hat{r}^c_t$. Model II characterizes an essential resource ($\underline{r}^c_t > 0$). The separability property of Model I is not found in Model II because the survival period T is an endogenous variable. The author showed that in this case resource consumption will be decreasing over time and its future value shadow price relative to the composite consumption good will increase at a discount rate during an intermediate stage.

Rawls' [69] principle of justice led Solow [81] to formulate his model such as to maximize the worst possible consumption level per head that can be maintained over an infinite planning horizon. It is then optimal that consumption of the composite good be constant throughout all generations. Under a Cobb-Douglas technology the author has been able to show that a solution exists for the simplest case of zero population growth and zero technical progress (Model I) if capital is a more important input than the exhaustible resource to the production process ($1-g-h > h$). The conclusions derived from Model I parallel DasGupta and Heal results. The author conjectures that Model II has a solution with a

higher consumption per head than Model I. It is also shown that in Model III no positive level of consumption per head can be maintained forever. The most general case, Model IV, proves difficult to explore and is left unresolved.

Stiglitz [84] introduces as his criterion the discounted sum of the total labor force utility defined as the sum of homogeneous individual workers' utility functions. The essentiality of the depletable resource is reflected by a Cobb-Douglas technology with exponential technological progress. For the special case of iso-elastic individual utility functions he has shown that under certain feasibility conditions this economy will approach a steady state where the resource will be utilized at a constant ratio of the remaining resource stock S_t. The rate of growth of consumption of the composite good and the savings rate will also be constant in this steady state.

1. II. 3. Sectoral Models

Sectoral models are characterized as optimization models that allocate primary resources and productive activities to fulfill the demand for end-use goods. The criterion mostly observed is the minimization of costs. Relative to primary supplies, sectoral models can be regarded as demand models as opposed to the supply models of section 1.II.1. Substitution possibilities are included more explicitly in sectoral models than in growth models where the fundamental substitution is between the depletable resource and the aggregate consumption good (capital). The practical importance of sectoral models is that they are designed with sufficient realism to be implemented with real data. Of course, the realism is

attained at the expense of the nice structural results that a high level of aggregation permits. Depletable resources sectoral models do not address explicitly the scarcity issue because they fail among other things to consider the problems of resource ownership. The intertemporal versions of sectoral models have disregarded the issue of ownership of the depletable resource as resource reserves constraints are imposed directly on the sector's operations.

The Brookhaven Energy System Optimization Model (BESOM) of Hoffman [42] in its original form is a static model of the energy sector designed primarily for the evaluation of energy technologies and policies. When demand can be met in more than one way from alternate supplies, BESOM will provide the policy that minimizes the costs of operating the energy sector. Each activity is characterized by an energy resource i, a conversion technology j and an end-use k. The flows x_{ijk} are the decision variables. Associated with the flows there are losses of two categories: upstream from the resource to conversion with efficiency a^s_{ij} and downstream from conversion to end-use with efficiency a^d_{jk}. The end-use demands d_k and the energy resources supplies s_i are given exogenously. BESOM is formulated as a linear program to

$$\text{minimize} \sum_{i,j,k} c_{ijk}\, x_{ijk}$$

$$\text{subject to} \sum_{i,j} a^d_{jk}\, x_{ijk} = d_k$$

$$\sum_{j,k} a^s_{ij}\, x_{ijk} \le s_i$$

plus capacity constraints, and environmental constraints,

$$x_{ijk} \ge 0,$$

where the c_{ijk} are the costs of utilizing devices in the path (i,j,k). Other constraints such as peaking and balancing load equations are included to model the electrical subsector. The dynamic characteristics of the energy system were to be introduced by applying the model sequentially. A major source of criticism to BESOM is the simple, perfectly inelastic, demand and supply functions.

Recognition that a static model of the energy system did not reflect the limited availability of energy resources led to a reformulation of BESOM as a Dynamic Energy System Optimization Model (DESOM) [62] in a multiperiod linear programming framework. Except for the inclusion of constraints on capacity and resource availability, the basic structure of BESOM is unchanged. Further indexing the exogenous data and the flow variables on a time factor DESOM is formulated so as to

$$\text{minimize} \sum_{i,j,k,t} \alpha^t c_{ijkt}\, x_{ijkt} + \sum_{i,t} \alpha^t p_{it}\, r_{it} + \sum_{i,j,t} \alpha^t q_{ijt}\, y_{ijt}$$

$$\text{subject to} \sum_{i,j} a^d_{jkt}\, x_{ijkt} = d_{kt} \tag{1.3a}$$

$$\sum_{j,k} a^s_{ijt}\, x_{ijkt} \le r_{it} \tag{1.3b}$$

$$\sum_t r_{it} \le S_{oi} \tag{1.3c}$$

$$r_{it} \le (1 + b_{it}) r_{i,t-1} \tag{1.3d}$$

plus capacity constraints, and

$$x_{ijkt},\ r_{it},\ y_{ijt} \ge 0.$$

In DESOM the supplies of energy resources are endogenous variables available at a price p_{it} per unit of i at time t. The capacity installed

in period t in the supply-conversion path (i,j) is y_{ijt} and costs q_{ijt} per unit. While constraints (1.3a) and (1.3b) are basically unchanged from BESOM, constraints (1.3c) and (1.3d) restrict the total utilization of resource i over time to S_{oi} and the rate of growth in resource usage to b_{it}. DESOM minimizes the present value of total costs of operating the energy sector during the planning horizon, with discount factor $\alpha(<1)$.

The allocation of energy resources over time, space and different categories of output that minimizes the discounted costs of meeting the demand for final products is determined in Nordhaus' [66] intertemporal sectoral model. The dynamic framework permitted the inclusion of constraints reflecting the limited availabilities of the exhaustible resources in the different areas. The problem formulated as an intertemporal linear program is

$$\text{minimize} \quad \sum_{i,j,k,l,t} c_{ijklt}\, x_{ijklt}$$

$$\begin{array}{lll} & & \underline{\text{shadow prices}} \\ \text{subject to} & \displaystyle\sum_{k,l,t} \frac{x_{ijklt}}{a_{jl}} \le S_{oij} & \lambda_{ij} \\ & \displaystyle\sum_{i,j} x_{ijklt} \ge d_{klt} & p_{klt} \\ & x_{ijklt} \ge 0 & \end{array}$$

where i, j, k, l and t denote respectively the country where the resource is located, the kind of resource, the country demanding the final product, demand categories and time period. The discounted unit costs c_{ijklt}, initial resource stocks S_{oij}, demands d_{klt} and the thermal efficiency coefficients a_{jl} are given exogenously. The decision variables are the

flows x_{ijklt}.

Letting $\lambda_{ij} \geq 0$ and $p_{klt} \geq 0$ denote the shadow prices to the first and second set of constraints respectively, we obtain from the optimality conditions that when $x_{ijklt} > 0$

$$c_{ijklt} + (\lambda_{ij}/a_{jl}) - p_{klt} = 0 \qquad (1.4)$$

must hold. Interpreting the shadow prices p_{klt} as the present value of the prices for final products that would prevail in a competitive economy in country k for demand category l at time t, (1.4) is a restatement of the fact that these prices not only reflect marginal costs but also a scarcity factor (royalty) adjusted for the efficiency of the conversion process. Experience with the above model showed that except for petroleum and coal the prices p_{klt} that resulted were close approximations to observed market prices, which makes the simplicity of the model quite attractive.

A more sophisticated model of the U. S. energy sector was devised by Manne [61] in his work in Energy Technology Assessment (ETA). The latter introduces finite elasticities for the demand for final products at the expense of a non-linear objective. The ETA model allocates a budget B among consumption divided into three categories: electric energy, non-electric energy and other items. Letting q_1 and q_2 denote the amount consumed of electric and non-electric sources and $c(q_1,q_2)$ represent the cost of fulfilling demand at levels q_1 and q_2, the maximand is a sum of discounted benefits. In any time period the benefits are given by

$$a q_1^{b_1} q_2^{b_2} + (B - c(q_1, q_2)) \tag{1.5}$$

where the first term represents the benefits derived from consumption of electric and non-electric energy, under the assumption of a unit elasticity of substitution, and the second term the benefits from expenditures in other items, under the assumption of constant marginal utility. The objective in (1.5) can be seen as the sum of consumers' and producers' surplus. A second interpretation of the objective is to minimize the negative of (1.5), which except for constant terms represents the total costs for the society of the energy supply demand conservation and of the interfuel substitution. The constraints in this model are linear and include, besides resource reserves, constraints of the type (1.3c), capacity constraints, bounds on new capacity introduction rates and aggregation of demand for final and intermediate items.

1. II. 4. Equilibrium Models

Equilibrium models are not explicit optimization models because no optimality criterion is specified a priori but are instead an ensemble of postulated relations that must hold if equilibrium is to be achieved. Shapiro [79] discusses cases where a well defined set of equilibrium conditions may be identified with the Kuhn-Tucker conditions for an underlying optimization model. The equilibria solutions can then be obtained as the solutions to the derived optimization problem. In general, fixed-point methods, as proposed by Scarf [73] can be used to determine equilibria whether or not they arise from implied optimization models.

Kennedy's [50] model of the world oil market falls in the former

category. Long-run equilibrium in the world oil market is characterized by a linear demand equation (1.6a), an economic constraint that no activity can make excess profits (1.6b), a material balance equation (1.6c) and a no-operation condition for those activities whose gross revenue fails to recover economic costs (1.6d). In matrix notation, let d denote the demands for final products, p the price for the products, c the unit costs of running the associated activity at unit level, x the level of operations and A the technology matrix of refining and transportation activities. Then the principles of economic equilibrium dictate that

$$p = Bd + b \tag{1.6a}$$

$$A'p \geq c \tag{1.6b}$$

$$Ax = d \tag{1.6c}$$

$$(c + A'p)x = 0 \tag{1.6d}$$

with

$$t \geq 0$$

$d_i \leq 0$ if i corresponds to a refined product

$d_i \geq 0$ if i corresponds to crude oil.

The author identified (1.6a), (1.6b), (1.6c) and (1.6d) as optimality conditions to the quadratic program

$$\text{maximize } b'd + \frac{1}{2} d'Bd - c'x$$

$$\text{subject to } Ax = d$$

$$d \geq 0, \quad x \geq 0$$

which is interpreted as a maximization of consumers' and producers' sur-

plus. Shapiro [77] suggested a dynamic reformulation of the above model that does not include resource stock constraints of the form (1.3c) explicitly.

1. II. 5. Econometric Models

The last class of models to be described as representative of an effort to model depletable resources are econometric models. The natural resources econometric models found in the literature are not general models but instead rather specific ones. They incorporate to different levels of refinement market and production characteristics as well as substitution effects that are peculiar to each individual resource.

In formulating a depletable resource econometric model the treatment of depletion becomes a central issue. Fisher, Cootner and Baily [21] in their econometric model of the world copper industry do not make explicit reference to depletion effects in their price and output relationships. In MacAvoy and Pindyck's [58] model of natural gas a "depletion" index of estimated potential reserves is introduced to capture the effects of exhaustion in the resource base in the rate of discoveries of natural gas. Also cumulative production of natural gas and oil are taken to be determinants of current natural gas production rates. The coal industry depletion model developed by Zimmerman [97] incorporates the depletion effects by estimating a "cumulative" cost function for underground coal mining that describes the changes in the incremental cost of coal as output cumulates over time. Coal reserves are assumed in this derivation to have a lognormal distribution.

1. III. Scope of the Thesis

After describing aspects of the problems posed by exhaustible resources extraction and reviewing some of the relevant efforts in depletable resource modelling, we proceed to describe and relate to previous work the approach we follow here.

In this thesis, we develop and analyze normative models of depletable resources. The major contributions stem from the integration of supply and sectoral models, whose normative aspects are inherent to their mathematical programming formulations. Mathematical programming concepts, particularly decomposition methods, facilitate the integration of these models. The decomposition approach provides a new framework in which to study the interactions between economic agents who supply and demand depletable resources. Moreover, the computational capabilities of mathematical programming and decomposition methods permit the integration of data-based normative and empirical models.

We approach the problem of depletable resource allocation over time in the traditional framework of supply and demand. On the demand side, we visualize the economy as divided into sectors. The economic sectors demand the depletable resource and, together with other inputs, engage in the production of end-use goods. The resource owners or suppliers, on the other side, are induced to extract it and eventually exhaust their reserves, by the revenue received from sales to the sectors. An equilibrium problem is defined once the behavior of these two actors in the depletable resource markets is postulated. It is in the modelling of this equilibrium and its solution methods that we concentrate the major

part of this thesis. What distinguishes our equilibrium framework from previous equilibrium models such as those discussed in section 1.II.4 is the existence of two active participants in the market and its intertemporal formulation, thereby permitting a more direct economic analysis of fixed resource stocks.

The economic sectors, represented by sectoral models, have the prescribed behavior of cost minimization. In each period, given a certain availability of the resource, they minimize the cost of meeting the consumers' demand for end-use goods. As compared to previous dynamic sectoral models such as the ones discussed in section 1.II.3., reserve constraints do not impose themselves directly into the sector activities. Instead, when the resource is privately owned, the fixed stock of the resource is an intertemporal supply constraint.

The modelling of the supply side of the economy parallels previous supply models (see section 1.II.1.) in the sense that we are interested in the behavior of the individuals or firms that own the resource. They are postulated to maximize the present value of profits over a planning horizon. However, in contrast to previous supply models, the time path of prices or demand functions for the depletable resource are not forecasts but derived normatively from the sectoral models.

Equilibrium between the economic sectors and the depletable resource suppliers is defined by linking the suppliers' revenue to the sectoral cost-savings in meeting demand for end-use goods.

In Chapter 2, we introduce the basic equilibrium model for a one-sector-one-supplier economy. A convergent iterative scheme is proposed

to solve for equilibrium. The algorithmic methods proposed for solution use techniques for non-differentiable optimization which are required when the sectoral model is a linear programming problem. This seems to be the general approach to sectoral modelling (see section 1.II.3.). The existence of an underlying optimization problem relates the economic equilibrium conditions with mathematical programming optimality conditions. Straightforward extensions in this framework are also covered. These include, among others, the existence of institutional agents (e.g., price controls, cost of externalities and tax depletion allowance) in the economy and the effect of the essentiality of the depletable resource to the sector operations. Computational results are given for a numerical example derived from BESOM [3].

The implications for intertemporal equilibrium of the existence of multiple economic sectors competing in each period for the depletable resource supply is part of the discussion in Chapter 3. The case of multiple depletable resource suppliers is also examined. The supply market structure ranges from the collusive behavior to the purely competitive case. The consideration of multiple depletable resources in the economy introduces several difficulties in our framework, mainly because the resources may be either substitutes or complements in different sectors' productive activities. Several alternatives are proposed and examined. The development of an integrated multiple economic sectors, multiple depletable resources and multiple suppliers equilibrium model is the goal of that chapter.

In Chapter 4 we reconsider, in the basic framework of a unique sec-

tor and a unique economic sector, the effect of the uncertainties associated with depletable resource exhaustion. A model with probabilistic technological transition in the sector's operations is developed. Uncertainty in end-use demands is handled by means of recourse programming and as a result it will bias downward the supplier's revenue. On the supply side, uncertainty in reserves and in each period's supply are discussed. A Markovian decision supplier's problem is proposed to model supply uncertainty.

Chapter 5 examines the case of the infinite planning horizon. Convergent finite-horizon approximations are proposed to solve for the infinite-horizon equilibrium in a stationary setting. Dynamic programming recursions will facilitate the successive approximation methods. A model combining a transient stage followed by a stationary stage is also presented and an iterative scheme is suggested as the solution procedure. The case of sectoral ownership or control over the depletable resource reserves over an infinite planning horizon is also considered. Well known properties of Leontief substitution systems are explored in the latter case.

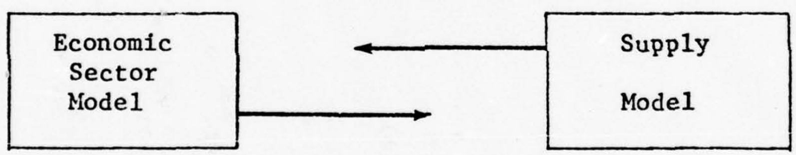
Lastly, Chapter 6 will introduce a more general equilibrium model, where the demand for end-use goods and the supply of alternative primary inputs are allowed to be elastic. The combination of econometric submodels with an optimization sectoral model and an optimization depletable resource supply model is proposed. The special case of linear demand and supply functions generates a quadratic programming sectoral problem which is treated by the development of quadratic parametric programming. Other

extensions are also discussed to different extents. Finally, the contribution of this thesis is summarized in Chapter 7.

Chapter 2: Equilibrium Model Between An Economic Sector and A Supplier of a Depletable Resource

2.I. Introduction

In this chapter we introduce an equilibrium model between an economic sector that is the sole user of a depletable resource and a unique supplier of this resource. Mathematical programming decomposition methods are used to study the interaction between a dynamic model of the economic sector and a supply model of this resource. The iterative scheme is illustrated below



Computational experience is given for a specific numerical example derived from the Brookhaven Energy System Optimization Model [BESOM(3)]. This chapter also sets the framework for the subsequent chapters that elaborate upon several aspects of the sectoral model, the supply model and the decomposition scheme.

2.II. Statement of the Basic Model

In a hypothetical economy we assume that the total stock of a depletable resource R is owned by a single individual (or firm), the supplier. In order to decide on a supply schedule $r^1, r^2, \ldots, r^T$ over a planning horizon of T periods, the supplier maximizes the net present value of profits resulting from the selling decisions. The value of the holdings of the R

units of the depletable resource $V_T(R)$ from period 1 through T is obtained by solving the *supplier's problem:*

$$V_T(R) = \max \sum_{t=1}^{T} \alpha^{t-1}\left\{\beta^t(r^t) - g^t\left(\sum_{j=1}^{t-1} r^j, r^t\right)\right\} + \alpha^T \beta^{T+1}\left(R - \sum_{t=1}^{T} r^t\right)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r^t \leq R \qquad (2.1)$$

$$r^t \geq 0 \qquad t = 1,2,\ldots,T$$

where $\alpha(<1)$ is a discount factor, β^t is the total revenue function, g^t is the extraction cost function and β^{T+1} is the salvage function. The revenue function denotes the supplier's revenue in period t for an amount r^t supplied of the depletable resource. It is assumed to be concave, continuous, non-decreasing and satisfying $\beta^t(0) = 0$. The extraction cost function is assumed convex, continuous, increasing and satisfying $g^t(\cdot,0) = 0$. It is a function of how much is extracted in period t as well as of the cumulative past extractions. The convex, increasing shape of the curve reflects increasing marginal costs resulting from increasing effort necessary to extract deeper deposits [e.g. Zimmerman(97)]. The salvage function is assumed to be concave, continuous, increasing and satisfying $\beta^{T+1}(0) = 0$. It gives a valuation for the resource stock left at the end of the planning horizon. It can be specified more precisely in a number of ways including infinite horizon approximations. This specification of the salvage function is discussed later in this chapter and in Chapter 5. Alternative specifications of the salvage function would result from contractual clauses and arbitrary resource owners' stock valu-

ations.

The objective is concave due to the concavity and convexity assumptions about the revenue, salvage and extraction cost functions. If β^t and g^t are differentiable, the Kuhn-Tucker conditions are necessary and sufficient conditions for optimality. Namely, they require the existence of a pair $(\hat{r},\hat{\lambda})$ satisfying

$$\frac{d\beta^t}{dr^t} - \frac{dg^t}{dr^t} - \sum_{i=t+1}^{T} \alpha^{i-t} \frac{dg^i}{dr^t} - \hat{\lambda}\alpha^{1-t} + \alpha^{T-t+1} \frac{d\beta^{T+1}}{dr^t} \leq 0$$

$$t = 1,2,\ldots,T \tag{2.2a}$$

$$\left(\frac{d\beta^t}{dr^t} - \frac{dg^t}{dr^t} - \sum_{i=t+1}^{T} \alpha^{i-t} \frac{dg^i}{dr^t} - \hat{\lambda}\alpha^{1-t} + \alpha^{T-t+1} \frac{d\beta^{T+1}}{dr^t}\right)\hat{r}^t = 0$$

$$t = 1,2,\ldots,T \tag{2.2b}$$

$$\sum_{t=1}^{T} \hat{r}^t \leq R, \qquad \hat{r}^t \geq 0 \qquad t = 1,2,\ldots,T \tag{2.2c}$$

$$\hat{\lambda}\left(\sum_{t=1}^{T} r^t - R\right) = 0 \tag{2.2d}$$

$$\hat{\lambda} \geq 0 \tag{2.2e}$$

In our development below, it will not always be the case that β^t and g^t are differentiable. In particular, the revenue function β^t when derived from a dynamic linear programming model will be concave, continuous but not everywhere differentiable. Technical difficulties due to nondif-

ferentiability can be overcome by the suitable use of subgradients, which exist everywhere, in place of the gradients in these conditions [e.g. Grinold (30)].

In order to interpret (2.2a) and (2.2b) it is insightful to verify that for $\hat{r}^t > 0$, (2.2b) can be rewritten as

$$\frac{d\beta^t}{dr^t} - \frac{dg^t}{dr^t} = \sum_{i=t+1}^{T} \alpha^{i-t} \frac{dg^i}{dr^t} + \hat{\lambda}\alpha^{1-t} - \alpha^{T-t+1} \frac{d\beta^{T+1}}{dr^t} \tag{2.3}$$

The left-hand side of (2.3) is the difference between marginal revenue and marginal immediate cost. The right-hand side can be regarded as a measure of total user's cost (see section 1.II.1.). The sacrifice of future profits due to present sales is composed of three items. The first term reflects the discounted effects of present extractions in elevating the cost of future extractions. The second term is a pure scarcity factor similar to Scott's [75], Gordon's [28] and Cummings' [10] measure of user's cost. Finally the last term reflects the sacrifice in salvage value. Therefore (2.3) can be restated as a requirement that marginal revenue equals marginal total costs (immediate plus total user's costs). The conditions (2.2c) are the constraints to the supplier's problem (2.1).

The shadow price of the depletable resource constraint λ, which can be interpreted as the marginal present value of incremental reserves is non-negative by (2.2e). By the complementary slackness condition (2.2d), it vanishes when the resource is not fully utilized over the planning horizon T.

The supplier's problem (2.1) can be reformulated as a dynamic pro-

gramming problem. There are two alternative state variables available: the reserve stock S and the accumulated extraction E. These two state variables are related through

$$S^t + E^t = R \qquad t = 1,2,\ldots,T.$$

Let the reserve stock S be the state variable and for $0 \leq S \leq R$ define the function

$$V_T^t(S) = \text{maximum present value of selling decisions from period } t \text{ through } T \text{ when } S \text{ is the stock of the depletable resource by the start of period } t. \tag{2.4}$$

The functions V_T^t satisfy the recursions

$$V_T^t(S) = \underset{0 \leq r \leq S}{\text{maximum}} \left\{ \beta^t(r) - g^t(R - S, r) + \alpha V_T^{t+1}(S - r) \right\} \tag{2.5}$$
$$t = T, T-1, \ldots, 1$$

where

$$V_T^{T+1}(S) = \beta^{T+1}(S) \qquad \text{for all } S.$$

The supplier's problem is solved by computing

$$V_T(R) \equiv V_T^1(R).$$

More details about this dynamic programming are given in section 2.IV.

The depletable resource is used by a single economic sector for

which vectors of end-use demands $\bar{d}^t$ are given for T periods. The demand is met by production transforming the primary supplies into end-use goods. Demand met through conversion and transmission of primary supplies in the energy sector (see section 1.II.3.) is an example. We assume the existence of a single depletable resource among the primary supplies available to the economic sector. Denoting by r the amount of the depletable resource available to the sector for production of end-use goods, a model of the economic sector can be developed to calculate in each period the *cost function*,

$$\Phi^t(r) = \text{minimum cost of meeting the demand for end-use goods in period } t \text{ when } r \text{ is the quantity of depletable resource available in period } t. \tag{2.6}$$

It is desirable that Φ^t be non-increasing in r and convex. This will be the case when the sectoral model is a linear program or the more general convex program formulated below. For all purposes at each fixed r, Φ^t can be identified in each period with a short-run cost function, where the amount of depletable resource r is fixed and given exogenously. However, Φ^t as a function of t incorporates long-run changes such as capacity expansions and new technologies. This is further discussed in Chapter 6. We further assume $\Phi^t(0) < +\infty$, or in other words, the demands in each period can be met (at finite cost) without using any of the depletable resource. If Φ^t is non-increasing in r, this is equivalent to assuming Φ^t

to be finite for all $r \geq 0$. The depletable resource is then inessential to the sector's operations in the sense of DasGupta and Heal [14]. The implications of the case when $\Phi^t(0) = +\infty$ will be explored in section 2.V.3. This assumption permits us to define the *cost-savings function*:

$$\Phi^t(0) - \Phi^t(r) = \text{cost savings in meeting the demand for end-use goods in period } t \text{ when } r \text{ is the quantity of depletable resource available in period } t. \tag{2.7}$$

If Φ^t is non-increasing, convex and finite, the cost-savings function (2.7) is non-negative, non-decreasing and concave. Figure 2.1 depicts the derivation of the cost savings function.

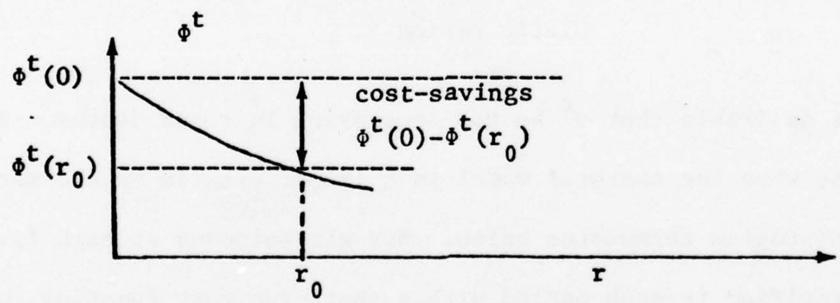


Figure 2.1

There is much choice in the determination of Φ^t, depending upon the sector's cost functions, the nature of the markets for alternative primary supplies, its production possibilities and substitution capabilities. We assume at this point that Φ^t is computed by solving in each period t a

sectoral problem. Different specifications of the sectoral problem and its implications are discussed in section 2.III.

With the above assumptions on the basic model, we proceed to determine how an equilibrium is reached between the depletable resource supplier and the economic sector in this hypothetical economy. We assume that the economic sector is willing to pay in period t for r units of the depletable resource a quantity not exceeding the cost savings implied from a resource availability of the amount r, $\phi^t(0)-\phi^t(r)$. Any quantity paid for a supply of r units less than $\phi^t(0)-\phi^t(r)$ lowers the sector's cost of meeting demand. The sector is indifferent when the cost of purchasing the supply is precisely $\phi^t(0)-\phi^t(r)$. This permits us to define the supplier's *revenue function*:

$$\beta^t(r) \equiv \phi^t(0) - \phi^t(r), \tag{2.8}$$

and to establish as an *equilibrium condition*:

"For any non-negative resource levels $r^1, r^2, \ldots, r^T$ satisfying $\sum_{t=1}^{T} r^t \le R$ we say that the supplier's and the sectoral problems are in *equilibrium* if these resource levels permit the supplier to maximize his profit; namely, (2.9)

$$V_T(R) = \sum_{t=1}^{T} \alpha^{t-1}\left\{\phi^t(0) - \phi^t(r^t) - g^t\left(\sum_{j=1}^{t-1} r^j, r^t\right)\right\} + \alpha^T\beta^{T+1}\left(R - \sum_{t=1}^{T} r^t\right).\text{"}$$

The revenue function β^t thus defined (2.8) is then concave and non-decreasing as desired.

Equilibrium between the economic sector and the unique supplier is reached through an iterative or tatônnement process depicted in Fig. 2.2: 1) the supplier announces a supply schedule of the depletable resource, $r^1, r^2, \ldots, r^T$; 2) the economic sector reacts to it by announcing for each period, at those levels, its cost of meeting demand Φ^t, or equivalently, its cost-savings through β^t as defined in (2.8), and its shadow price π^t which can be viewed as bid prices for marginal depletable resource units; 3) either equilibrium (2.9) has been reached or, given this new information, the supplier revises his supply schedule. Further details on this iterative process as well as its convergence properties are presented in section 2.III.

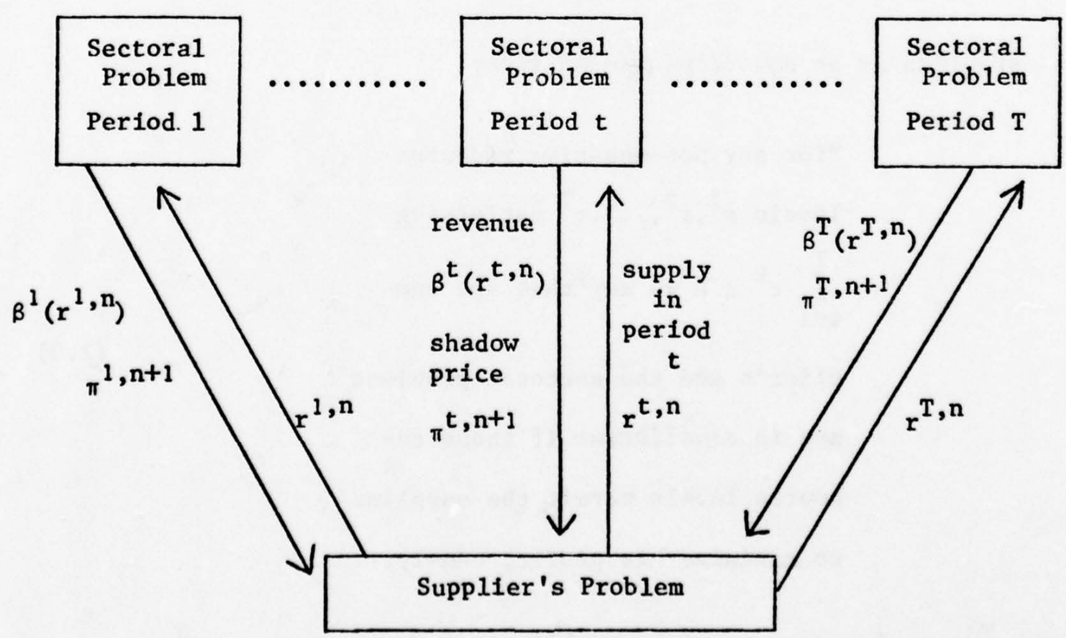


Figure 2 .2

2.III. Convergence to Equilibria

In this section we illustrate the computation of Φ^t for two different specifications of the sectoral problems: linear and convex. Convergence properties differ in both situations and are summarized in Theorems 2.1 and 2.2, respectively.

2.III.1-A LP-Sectoral Problem

The functions Φ^t are computed by solving in each period t the *LP-sectoral problem*:

$$\Phi^t(r) = \min\ c^t x^t + f^t s^t \tag{2.10a}$$

$$\text{s.t. } \rho^t x^t \leq r \tag{2.10b}$$

$$A_1^t x^t - s^t \leq 0 \tag{2.10c}$$

$$A_2^t x^t \geq d^t \tag{2.10d}$$

$$0 \leq x^t \tag{2.10e}$$

$$0 \leq s^t \leq \mathcal{S}^t \tag{2.10f}$$

where the vector decision variables x^t and s^t denote the activities level of operation and the amounts purchased of the alternative primary supplies respectively. Unit costs are given by the non-negative vectors c^t and f^t respectively. The matrices A_1^t and A_2^t give the technological coefficients for supply and demand respectively and ρ^t is a vector denoting the unit depletable resource usage of the corresponding activities. Since the depletable resource is non-producible, $\rho^t \geq 0$. The objective

(2.10a) is the minimization of costs in each period t. Costs include both the operational costs of the sector's activities as well as the expenditures on the alternative primary supplies' purchase. The fact that the depletable resource usage cannot exceed its availability is indicated by (2.10b). Constraints (2.10c) and (2.10d) are the ordinary supply and demand restrictions. Finally (2.10e) and (2.10f) represent the non-negativity constraints as well as the existence of upper bounds on the utilization of alternative primary supplies. The sectoral model BESOM [3], described in section 1.II.3., is exactly of this form, except for environmental constraints which are not binding.

The functions ϕ^t derived from (2.10) have desirable properties as summarized by the following Lemma.

<u>Lemma 2.1</u> ϕ^t given by (2.10) is

i) non-increasing and convex;

ii) piecewise linear with a finite number of segments.

Proof: Let $X^t(r)$ denote the set of feasible solutions to (2.10) or equivalently

$$X^t(r) = \{(x^t,s^t)\,|\,\rho^t x^t \le r, A_1^t x^t - s^t \le 0, A_2^t x^t \ge d^t, x^t \ge 0, 0 \le s^t \le \bar{s}^t\} .$$

It is trivial to observe that for $r_1 \geq r_2, X^t(r_2) \subseteq X^t(r_1)$ and therefore it must be that

$$\phi^t(r_1) \le \phi^t(r_2) .$$

To prove convexity, notice that if $\{x_1^t, s_1^t\} \in \mathcal{X}^t(r_1)$ and $\{x_2^t, s_2^t\} \in \mathcal{X}^t(r_2)$ then

$$\left\{\omega x_1^t + (1-\omega)x_2^t, \omega s_1^t + (1-\omega)s_2^t\right\} \in \mathcal{X}^t\left(\omega r_1 + (1-\omega)r_2\right)$$

for any $0 \le \omega \le 1$. Letting these be optimal solutions associated with $\Phi^t(r_1)$ and $\Phi^t(r_2)$ respectively, we have

$$\Phi^t\left(\omega r_1 + (1-\omega)r_2\right) \le c^t[\omega x_1^t + (1-\omega)x_2^t] + f^t[\omega s_1^t + (1-\omega)s_2^t] =$$

$$\omega\Phi^t(r_1) + (1-\omega)\Phi^t(r_2)$$

which completes the proof of (i).

To prove (ii), consider determining

$$\Phi(b) = \min cx$$

$$\text{s.t. } Ax = b,\ x \ge 0.$$

Problem (2.10) can be easily rewritten in this format. For any b, let B denote the optimal basis and partition $A = [B,N]$, $c = [c_B, c_N]$ and $x = [x_B, x_N]$. By letting $x_N = 0$ and $x_B = B^{-1}b$, the optimal value for Φ is given by

$$\Phi(b) = c_B B^{-1} b.$$

This solution remains optimal as long as $x_B = B^{-1}b \ge 0$. Thus for all b within this polyhedron Φ is linear in b as given above. With each basis there is a unique linear segment of Φ associated with it. Since the number of bases that can be selected from A is finite there can only be a

a finite number of such segments.||

Let π denote the shadow prices or dual variables on the depletable resource availability constraint (2.10b), v denote the shadow prices on the supply constraints (2.10c), u denote the vector of shadow prices on the demand constraints (2.10d), and w denote the shadow prices on the upper-bound constraints (2.10f). Linear programming duality theory gives the cost-savings function as

$$\beta^t(r) \equiv \Phi^t(0) - \Phi^t(r) = \underset{k=1,\ldots,K^t}{\text{minimum}} \left\{ \left(\Phi^t(0) - u^{t,k}d^t + w^{t,k}s^t\right) + \pi^{t,k}r \right\} \tag{2.11}$$

where $K^t = \left\{\left(\pi^{t,k},\ v^{t,k},\ u^{t,k},\ w^{t,k}\right)\right\}$ is the set of extreme points of the dual problem

$$\begin{aligned} \Phi^t(r) = \max \ -\pi^t r \qquad & - w^t s^t + u^t d^t \\ -\pi^t \rho^t - v^t A_1^t \qquad & + u^t A_2^t \le c^t \\ v^t \qquad - w^t \qquad & \le f^t \\ \pi^t \ge 0,\ v^t \ge 0,\ u^t \ge 0,\ & w^t \ge 0 \end{aligned} \tag{2.12}$$

with cardinality K^t finite as a result of Lemma 2.1.

From a parametric linear programming viewpoint as r is increased from 0, the function $\Phi^t(0) - \Phi^t(r)$ can be calculated in terms of a sequence of decreasing π values which are optimal in problem (2.12). The form of $\Phi^t(0) - \Phi^t(r)$ as given by (2.11) is depicted in Figure 2.3.

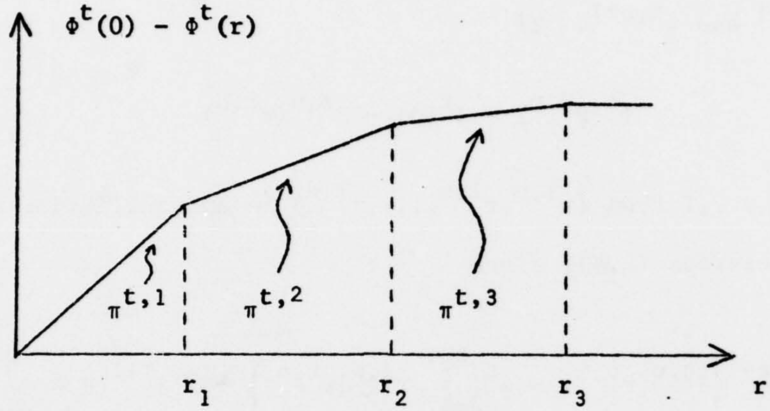


Figure 2.3

An iterative approach to solve the supplier's problem (2.1) is required because in general the concave piecewise linear functions β^t generated by (2.10) and (2.8) are composed of a large number of linear segments. This difficulty is overcome by working with the upper-bound approximations

$$\tilde{\beta}^{t,n}(r) = \underset{k=1,\ldots,L^{t,n}}{\text{minimum}} \left\{(\Phi^t(0) - u^{t,k}d^t + w^{t,k}s^t) + \pi^{t,k}r\right\} \tag{2.13}$$

which results when some linear segments are omitted from (2.11) (i.e. $L^{t,n} \leq K^t$). By upper-bound approximations we mean $\tilde{\beta}^{t,n}(r) \geq \beta^t(r)$ for all $r \geq 0$. The approach is depicted in Figure 2.2.

At iteration n, the supplier's problem (2.1) is solved using the approximation $\tilde{\beta}^{t,n}$. The result is a vector of non-negative resource levels $r^{t,n}$ satisfying $\sum_{t=1}^{T} r^{t,n} \leq R$ which are sent to the sectoral problems (2.10). The sectoral problems are optimized using these resource

levels yielding the costs $\phi^t(r^{t,n})$ and the dual variables $u^{t,n+1}$, $\pi^{t,n+1}$, $w^{t,n+1}$ and $v^{t,n+1}$. If

$$\phi^t(r^{t,n}) = \phi^t(0) - \tilde{\beta}^{t,n}(r^{t,n})$$

for all $t=1,2,\ldots,T$ then $(r^{1,n},r^{2,n},\ldots,r^{T,n})$ is an equilibrium solution because it satisfies (2.9), since

$$V_T(R) \le \sum_{t=1}^{T} \alpha^{t-1}\left\{\tilde{\beta}^{t,n}(r^{t,n}) - g^t\left(\sum_{j=1}^{t-1} r^{j,n}, r^{t,n}\right)\right\} + \alpha^T\beta^{T+1}\left(R - \sum_{t=1}^{T} r^{t,n}\right) \tag{2.14a}$$

and,

$$V_T(R) \ge \sum_{t=1}^{T} \alpha^{t-1}\left\{\phi^t(0) - \phi^t(r^{t,n}) - g^t\left(\sum_{j=1}^{t-1} r^{j,n}, r^{t,n}\right)\right\} + \alpha^T\beta^{T+1}\left(R - \sum_{t=1}^{T} r^{t,n}\right). \tag{2.14b}$$

Conversely if

$$\phi^t(r^{t,n}) > \phi^t(0) - \tilde{\beta}^{t,n}(r^{t,n})$$

for some $t=1,2,\ldots,T$ then the supplier has strictly overestimated the revenue that can be received from the economic sector in those periods. In the latter case the supplier reoptimizes using the tighter approximation $\tilde{\beta}^{t,n+1}$ in period t which results by adding a new linear segment to the approximation (2.13),

$$(\phi^t(0) - u^{t,n+1}d^t + w^{t,n+1}g^t) + \pi^{t,n+1}r$$

or by taking $r^{t,n}$ as a reference level,

$$\phi^t(0) - \phi^t(r^{t,n}) + \pi^{t,n+1}(r-r^{t,n}).$$

Equivalently,

$$\mathcal{L}^{t,n+1} = \mathcal{L}^{t,n} \cup \{u^{t,n+1}, v^{t,n+1}, w^{t,n+1}, \pi^{t,n+1}\}$$

where $\mathcal{L}^{t,n}$ is the subset of $\mathcal{K}^t$ known at iteration n, with cardinality $L^{t,n}$. The new resource levels, $r^{t,n+1}$, are then sent to the sectoral problems and the process is repeated.

This iterative approach will converge to an equilibrium solution (2.9) in a finite number of iterations. The argument is summarized in the following theorem.

Theorem 2.1: (Finite Convergence) The iterative scheme described above converges to an equilibrium solution after a finite number of iterations between the supplier's problem (2.1) and the LP-sectoral problems (2.10).

Proof: This is because each iteration between the LP-sectoral problems and the supplier's problem either finds an equilibrium solution or causes at least one linear segment to be added to the approximations (2.13) for some t. Since by Lemma 2.1 there is only a finite number of such segments an equilibrium must be reached in, at most, $\sum_{t=1}^{T} K^t$ iterations.||

The iterative approach described above can be seen equivalent to applying Benders' [2] decomposition to the problem

$$V_T(R) = \max \sum_{t=1}^{T} \alpha^{t-1}\{\Phi^t(0) - c^t x^t - f^t s^t - g^t(\sum_{j=1}^{t-1} r^j, r^t)\} + \alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} r^t)$$

$$\begin{aligned} \text{s.t. } \rho^t x^t - r^t &\le 0 \qquad t=1,2,\ldots,T \\ A_1^t x^t - \mathcal{s}^t &\le 0 \qquad t=1,2,\ldots,T \\ A_2^t x^t &\ge d^t \qquad t=1,2,\ldots,T \\ \sum_{t=1}^{T} r^t &\le R \\ 0 \le x^t,\ 0 \le s^t \le \mathcal{s}^t,\ 0 &\le r^t \qquad t=1,2,\ldots,T \end{aligned} \tag{2.15}$$

For a fixed supply schedule $r^1, r^2, \ldots, r^T$, problem (2.15) is separable in time. It breaks into T subproblems, the LP-sectoral problems (2.10). The master problem is the supplier's problem (2.1) under the definition (2.8) for the supplier's revenue function. Problem (2.15) can be interpreted as a maximization of the discounted benefit, under the assumption of constant marginal utilities with $\Phi^t(0)$ taken as reference value. The other interpretation is the minimization of the present value of sectoral plus supplier's costs.

As a matter of fact, when the unique economic sector considered here is an aggregate of a competitive demand market and the unique supplier's extraction cost function is an aggregate extraction cost function of a competitive supply market, problem (2.15) is a maximization of the dis-

counted sum of consumers' plus producers' surplus. This is because in the latter case the sectoral cost-savings as defined by (2.7) is a measure of consumers' surplus and the unique supplier is assumed through the definition (2.8) to capture as revenue the full amount of this surplus. Consequently the resource allocation resulting from solving (2.15) through the iterative approach described above will be efficient. We shall delay a further discussion of this matter until Chapter 3.

2.III.2. A Convex Sectoral Problem

In this subsection we illustrate the computation of Φ^t by solving in each period t a *convex sectoral problem*:

$$\Phi^t(r) = \min c^t(x^t) + f^t(s^t) \tag{2.16a}$$

$$\text{s.t. } F^t(d^t;\{x^t,s^t,z^t\}) \leq 0 \tag{2.16b}$$

$$z^t \leq r \tag{2.16c}$$

$$0 \leq s^t \leq \bar{s}^t \tag{2.16d}$$

$$0 \leq x^t,\ 0 \leq z^t \tag{2.16e}$$

where the triple $\{x^t,s^t,z^t\}$ denotes the inputs to the production of end-use goods: the non-primary resources, alternative primary supplies and the depletable resource, respectively. The objective (2.16a) is the minimization of costs where the cost function c^t and f^t are assumed convex continuous and increasing. Constraints (2.16b) give the feasible production possibilities. The vector-valued function F^t is assumed convex and continuous. The depletable resource usage is constrained by its availa-

bility by (2.16c). Upper bound restrictions and non-negativity are imposed by (2.16d) and (2.16e). We further assume that problem (2.16) satisfies a constraint qualification, or Slater condition, requiring that for all $r \geq 0$, there exists a triple $\{x^t, s^t, z^t\}$ satisfying (2.16c), (2.16d) and (2.16e) such that

$$F^t(d^t; \{x^t, s^t, z^t\}) < 0.$$

Under the above conditions we can show

Lemma 2.2 Φ^t given by (2.16) is non-increasing in r and convex.

Proof: Characterize the set of feasible solutions to (2.16) by

$$X^t(r) = \left\{\{x^t, s^t, z^t\} \mid F^t(d^t; \{x^t, s^t, z^t\}) \leq 0, 0 \leq z^t \leq r, 0 \leq x^t, 0 \leq s^t \leq \bar{s}^t\right\}$$

As in the proof of Lemma 2.1, it is easy to show that for $r_1 \geq r_2$, $X^t(r_2) \subseteq X^t(r_1)$. Therefore we must have

$$\Phi^t(r_1) \leq \Phi^t(r_2).$$

To prove convexity, we need observe that since F^t is convex, if $\{x_1^t, s_1^t, z_1^t\} \in X^t(r_1)$ and $\{x_2^t, s_2^t, z_2^t\} \in X^t(r_2)$, then

$$\{\omega x_1^t + (1-\omega)x_2^t, \omega s_1^t + (1-\omega)s_2^t, \omega z_1^t + (1-\omega)z_2^t\} \in$$

$$X^t(\omega r_1 + (1-\omega)r_2)$$

for any $0 \leq \omega \leq 1$. Letting these be the optimal solution for $\Phi^t(r_1)$ and

$\Phi^t(r_2)$ respectively we have, by the convexity of c^t and f^t, that

$$\Phi^t(\omega r_1 + (1-\omega)r_2) \le c^t(\omega x_1^t + (1-\omega)x_2^t) + f^t(\omega s_1^t + (1-\omega)s_2^t) \le$$

$$\omega\left[c^t(x_1^t) + f^t(s_1^t)\right] + (1-\omega)\left[c^t(x_2^t) + f^t(s_2^t)\right] =$$

$$\omega\Phi^t(r_1) + (1-\omega)\Phi^t(r_2)$$

which completes the proof of the lemma.||

Since Φ^t is a real-valued convex function it is lower semi-continuous and continuous in any open subset of its domain (see Rockafellar [70]). In order to guarantee continuity at r=0, we need require that the point-to-set mapping $X^t(r)$ be continuous at r=0 (see Meyer [64]). While this is not crucial for our development since the resulting revenue function β^t will be upper-semi-continuous, whenever we need the continuity of β^t we are making this assumption about the convex sectoral problem (2.16).

Let μ denote the shadow prices or multipliers on constraints (2.16b) and π denote the shadow prices on the depletable resource availability constraint (2.16c). Under the Slater condition, convex programming duality theory (see Lasdon [54]) gives the cost-savings function as:

$$\beta^t(r) \equiv \Phi^t(0) - \Phi^t(r) = \underset{\substack{\mu \ge 0 \\ \pi \ge 0}}{\text{minimum}} \{\Phi^t(0) - \Upsilon^t(r;\mu,\pi)\}$$

where

$$\gamma^t(r;\mu,\pi) = -\pi^t r + \min\left\{c^t(x^t) + f^t(s^t) + \right.$$

$$\left. \mu^t F^t(d^t;\{x^t,s^t,z^t\}) + \pi^t z^t\right\}$$

$$\text{s.t. } 0 \leq x^t,\ 0 \leq s^t \leq \mathcal{S}^t,\ 0 \leq z^t$$

An iterative approach is required to solve the supplier's problem (2.1) because generating β^t may involve doing a non-trivial parametric non-linear programming. This is also overcome by working with the upper-bound approximations:

$$\tilde{\beta}^{t,n}(r) = \underset{k=1,\ldots,L^{t,n}}{\text{minimum}} \{\phi^t(0) - \gamma^t(r;\mu^{t,k},\pi^{t,k}\} \tag{2.17}$$

which results when only a finite subset of the multipliers is considered (i.e., $L^{t,n} < \infty$). It is trivial to verify that the resulting approximation $\tilde{\beta}^{t,n}$ is a concave piecewise linear function. The approximations (2.17) are depicted in Figure 2.4.

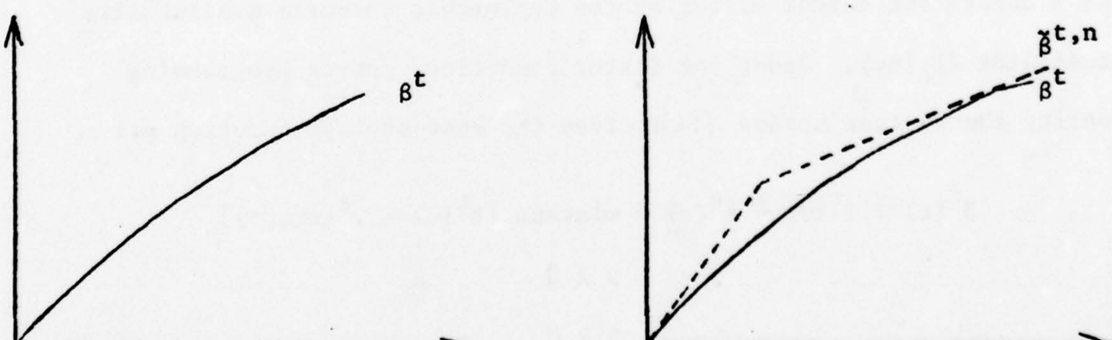


Figure 2.4

The iterative process is equivalent to the linear case. At iteration n, the supplier's problem is solved using the approximation $\tilde{\beta}^{t,n}$. The result is a vector of non-negative resource levels $r^{t,n}$ satisfying $\sum_{t=1}^{T} r^{t,n} \leq R$ which are sent to the sectoral problems (2.16). The sectoral problems (2.16) are optimized by any applicable method using these resource levels yielding costs $\phi^t(r^{t,n})$ and the multipliers $\mu^{t,n+1}$ and $\pi^{t,n+1}$. If

$$\phi^t(r^{t,n}) = \phi^t(0) - \tilde{\beta}^{t,n}(r^{t,n})$$

for all $t=1,2,\ldots,T$ then an equilibrium solution (2.9) has been reached. Conversely, if

$$\phi^t(r^{t,n}) > \phi^t(0) - \tilde{\beta}^{t,n}(r^{t,n})$$

for some $t=1,2,\ldots,T$, a tighter approximation in period t, $\tilde{\beta}^{t,n+1}$, is available for the supplier reoptimization by the inclusion of the new linear segment

$$(\phi^t(0) - \gamma^t(r;\mu^{t,n+1},\pi^{t,n+1}))$$

or by taking $r^{t,n}$ as a reference level,

$$\phi^t(0) - \phi^t(r^{t,n}) + \pi^{t,n+1}(r - r^{t,n}).$$

Equivalently,

$$L^{t,n+1} = L^{t,n} \cup \{\mu^{t,n+1},\pi^{t,n+1}\}.$$

New resource levels $r^{t,n+1}$ are sent to the sectoral problems and the process is repeated.

This iterative approach can be identified with applying Geoffrion's [26] generalized Benders' decomposition to the problem:

$$\max \sum_{t=1}^{T} \alpha^{t-1}\{\Phi^t(0) - c^t(x^t) - f^t(s^t) - g^t(\sum_{j=1}^{t-1} r^j, r^t)\} + \alpha^T\beta^{T+1}(R - \sum_{t=1}^{T} r^t)$$

$$\text{s.t.} \quad F^t(d^t;\{x^t,s^t,z^t\}) \le 0 \qquad t=1,2,\ldots,T$$

$$z^t \le r^t \qquad t=1,2,\ldots,T \tag{2.18}$$

$$\sum_{t=1}^{T} r^t \le R$$

$$0 \le x^t,\ 0 \le s^t \le \mathcal{s}^t,\ 0 \le z^t,\ 0 \le r^t \qquad t=1,2,\ldots,T$$

The statement of this equivalence as well as the interpretation of (2.18) are readily seen similar to that of problem (2.15).

<u>Theorem 2.2</u>: (Infinite Convergence) The iterative approach converges to an equilibrium solution between the supplier's problem (2.1) and the convex sectoral problems (2.16) as the number of iterations tends to infinity.

Proof: The proof of this theorem encompasses features of related proofs by Geoffrion [26] and Magnanti, Shapiro and Wagner [59].

Let V_T denote the equilibrium (2.9) value of the supplier's problem (2.1), and $< V_T^n;(r^{1,n},r^{2,n},\ldots,r^{T,n}) >$ the sequence of optimal solutions to the supplier's problem at successive executions of the iterative approach thus described. Similarly let $<(\mu^{1,n+1},\mu^{2,n+1},\ldots,\mu^{T,n+1});$ $(\pi^{1,n+1},\ldots,\pi^{2,n+1},\ldots,\pi^{T,n+1}) >$ denote the corresponding sequence of

shadow prices to the sectoral problem (2.16). Since

$$\beta^t(r) \le \tilde{\beta}^{t,n+1}(r) \le \tilde{\beta}^{t,n}(r) \qquad \text{for all } r$$

we observe that

$$V_T \le V_T^{n+1} \le V_T^n \qquad \text{for all } n=1,2,\ldots .$$

That is, the sequence $\{V_T^n\}$ is non-increasing and bounded from below. The sequence $\{r^{t,n}\}$ is in the compact set $[0,R]$ for $t=1,2,\ldots,T$. In addition, the Slater constraint qualification implies that the set of optimal multipliers $\langle\mu;\pi\rangle$ is uniformly bounded (see Geoffrion [26]). Thus, we may choose a converging subsequence, if necessary, such that

$$\lim V_T^n = \hat{V}_T$$

$$\lim r^{t,n} = \hat{r}^t \qquad t=1,2,\ldots,T$$

$$\lim \mu^{t,n} = \hat{\mu}^t \qquad t=1,2,\ldots,T$$

$$\lim \pi^{t,n} = \hat{\pi}^t \qquad t=1,2,\ldots,T$$

yielding

$$V_T \le \hat{V}_T \le V_T^n \qquad \text{for all } n.$$

In order to complete the proof we need to show now that $\hat{V}_T \le V_T$.

The following relation holds by the accumulation of linear segments in the approximations (2.17),

$$\tilde{\beta}^{t,n+1}(r^{t,n+1}) \le \phi^t(0) - \gamma^t(r^{t,n+1};\mu^{t,n},\pi^{t,n}) \qquad t=1,2,\ldots,T. \tag{2.19}$$

Subtracting from (2.19), the extraction cost $g^t(\sum_{j=1}^{t-1} r^{j,n+1}, r^{t,n+1})$ and after multiplication by α^{t-1}, adding over t and subsequently adding $\alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} r^{t,n+1})$ on both sides yields

$$V_T^{n+1} \le \sum_{t=1}^{T} \alpha^{t-1}\{\phi^t(0) - \gamma^t(r^{t,n+1};\mu^{t,n},\pi^{t,n}) - g^t(\sum_{j=1}^{t-1} r^{j,n+1}, r^{t,n+1}) +$$

$$\alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} r^{t,n+1}).$$

From the continuity of g^t, β^{T+1} and γ^t, taking limits on both sides gives

$$\hat{V}_T \le \sum_{t=1}^{T} \alpha^{t-1}\{\phi^t(0) - \gamma^t(\hat{r}^t;\hat{\mu}^t,\hat{\pi}^t) - g^t(\sum_{j=1}^{t-1} \hat{r}^j, \hat{r}^t)\} + \alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} \hat{r}^t)$$

We need to show that

$$\gamma^t(\hat{r}^t;\hat{\mu}^t,\hat{\pi}^t) = \phi^t(\hat{r}^t) \qquad t=1,2,\ldots,T$$

or, in other words, that $<\hat{u}^t;\hat{\pi}^t>$ belongs to the point-to-set mapping of optimal multipliers for $\phi^t(\hat{r}^t)$. For this purpose let this set be characterized by

$$\mathcal{U}^t(r) = \{<u^t;\pi^t>> 0 \,|\, \gamma^t(r;\mu^t,\pi^t) = \max \gamma^t(r;\mu_1^t,\pi_1^t)\}$$

$$\mu_1 \ge 0$$

$$\pi_1^t \ge 0$$

From the continuity of γ^t and Meyer's [(64) Theorem 1.5], $\mathcal{U}^t(r)$ is an an upper-semi-continuous point-to-set mapping and hence

$$<\hat{u}^t;\hat{\pi}^t> \in \mathcal{W}^t(\hat{r}^t) \qquad t=1,2,\ldots,T.$$

From Lemma 2.2, Φ^t is a real valued convex function and hence lower semi-continuous (see Rockafellar [70]). From the continuity of g^t, $t=1,2,\ldots,T$ and β^{T+1}, for an arbitrary $\varepsilon > 0$, there exists an N such that for $n \geq N$

$$\Phi^t(r^{t,n}) \leq \Phi^t(\hat{r}^t) + \frac{\varepsilon\ (1-\alpha)}{3\ (1-\alpha^T)} \qquad t=1,2,\ldots,T$$

$$g^t\left(\sum_{j=1}^{t-1} r^{j,n}, r^{t,n}\right) \leq g^t\left(\sum_{j=1}^{t-1} \hat{r}^j, \hat{r}^t\right) + \frac{\varepsilon\ (1-\alpha)}{3\ (1-\alpha^T)}$$

$$t=1,2,\ldots,T$$

and

$$\beta^{T+1}\left(R - \sum_{t=1}^{T} r^{t,n}\right) \geq \beta^{T+1}\left(R - \sum_{t=1}^{T} \hat{r}^t\right) - \frac{\varepsilon}{3\alpha^T}$$

yielding

$$V_T \leq \hat{V}_T \leq \sum_{t=1}^{T} \alpha^{t-1}\{\Phi^t(0) - \Phi^t(r^{t,n}) - g^t\left(\sum_{j=1}^{t-1} r^{j,n}, r^{t,n}\right)\} +$$

$$\alpha^T\beta^{T+1}\left(R - \sum_{t=1}^{T} r^{t,n}\right) + \varepsilon \leq V_T + \varepsilon.$$

The last step follows because $(r^{1,n}, r^{2,n}, \ldots, r^{T,n})$ is a feasible schedule to the supplier's problem (2.1).

Since ε was chosen arbitrarily, it must be that

$$\lim V_T^n = \hat{V}_T = V_T$$

which completes the proof.||

2.IV. Solution Methods for the Sectoral and Supplier Subproblems

In this section we shall briefly discuss possible solution methods to the sectoral and supplier's problems.

2.IV.1. The Sectoral Problems

Of course the LP-sectoral problems (2.10) are trivial to solve and require no further discussion. When the sectoral problem is a non-linear convex program of the form (2.16) it can be solved at each iteration by an approximating grid linearization, as suggested by Wolfe [93]. Linearization of the non-linear functions c^t, f^t and F^t on a set of grid points $\{(x_i^t, s_i^t, z_i^t)\}$ yields an equivalent linear program that results from restricting the decision variables to lie on the convex hull generated by the grid (see Lasdon [54]).

There is, however, no general rule for determining the number of grid points required for an accurate approximation of the non-linear functions, Dantzig's [12] generalized linear-programming would improve the approximations by generating iteratively new grid points. Convergence to an optimal solution has been proved for convex programs, such as the convex sectoral problem (2.16), to occur as the number of iterations tends to infinity. Once an approximating grid is considered satisfactory, possibly after a few generalized linear-programming iterations, the approximated cost-function $\tilde{\Phi}^t(r)$ is convex, non-increasing and piecewise linear with a finite number of segments. For the approximated (linearized) convex sectoral program finite convergence as established in Theorem 2.1 holds.

Hence, for all purposes, in some of the subsequent chapters we shall concentrate on piecewise linear cost-savings functions that result from either an LP-sectoral problem (2.10) or grid linearizations of convex sectoral problems such as (2.16).

2.IV.2. The Supplier's Problem

In the case of a non-linear supplier's problem in order to benefit from the simplicity of the linear constraints, algorithms equivalent to the reduced-gradient method of Wolfe [93] can be particularly useful. Alternative solution methods include solving the unconstrained supplier's problem with the salvage function extended to $(-\infty, +\infty)$

$$\beta^{T+1}(S) = \begin{cases} \beta^{T+1}(S) & S \geq 0 \\ -\infty & S < 0 \end{cases}$$

In practice this can be handled by penalty methods (see Luenberger [56]) which would set an arbitrarily large negative penalty for supply schedules violating the constraint $\sum_{t=1}^{T} r^t \leq R$, or in other words, with $\sum_{t=1}^{T} r^t > R$. Dual methods for solving the supplier's problem can also be of practical interest as the Lagrangian takes a simple form. In the latter case we would solve

$$\min_{\lambda \geq 0} \{+ \lambda R + \max_{\substack{r^t \geq 0 \\ t=1,2,\ldots,T}} \{ \sum_{t=1}^{T} [\alpha^{t-1}\{\beta^t(r^t) - g^t(\sum_{j=1}^{t-1} r^j, r^t)\} - \lambda r^t] +$$

$$\alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} r^t)\}\}.$$

Of special interest is the case of piecewise linear revenue, extraction cost and salvage functions, as the resulting supplier's problem becomes equivalent to a linear program. This is illustrated for the special case where

$$g^t(E,r) = e(E+r) - e(E) \qquad t=1,2,\ldots,T \tag{2.20a}$$

where e is a convex piecewise linear "cumulative" cost function (e.g. Zimmerman [97]), giving the total accumulated cost of extracting E units of the resource, independent of the time period. Let it be represented by

$$e(E) = \underset{j=1,2,\ldots,J}{\text{maximum}} \{e^j + \gamma^j E\} \tag{2.20b}$$

with $e^j \leq 0$ and $\gamma^j \geq 0$. For concave piecewise linear revenue and salvage functions

$$\beta^t(r) = \underset{k=1,2,\ldots,K^t}{\text{minimum}} \{b^{t,k} + \pi^{t,k} r\} \tag{2.21}$$

$$\beta^{T+1}(S) = \underset{k=1,2,\ldots,K^{T+1}}{\text{minimum}} \{b^{T+1,k} + \pi^{T+1,k} S\} \tag{2.22}$$

with $b^{t,k}$, $\pi^{t,k} \geq 0$, the supplier's problem (2.1) can be conveniently rewritten as a linear program with $\sum_{t=1}^{T+1} K^t + TJ$ additional constraints. The *LP-supplier's problem* is then

$$V_T(R) = \max \sum_{t=1}^{T} \alpha^{t-1} z^t + \sum_{t=1}^{T-1} \alpha^{t-1}(1-\alpha)y^t + \alpha^{T-1}y^T + \alpha^T z^{T+1}$$

$$\text{s.t.} \quad z^t \le b^{t,k} + \pi^{t,k} r^t \qquad k=1,2,\ldots,K^t, \quad t=1,2,\ldots,T$$

$$y^t \le -e^j - \gamma^j\left(\sum_{i=1}^{t} r^i\right) \qquad j=1,2,\ldots,J, \quad t=1,2,\ldots,T$$

$$z^{T+1} \le b^{T+1,k} + \pi^{T+1,k}\left(R - \sum_{i=1}^{T} r^i\right) \qquad k=1,2,\ldots,K^{T+1} \tag{2.23}$$

$$\sum_{t=1}^{T} r^t \le R$$

$$r^t \ge 0 \qquad t=1,2,\ldots,T.$$

We can notice that when the planning horizon T is long and the revenue functions β^t and cumulative cost function e have a large number of linear segments, problem (2.23) may turn into a large scale linear program. For this purpose we illustrate the solution to the supplier's problem through the sequence of backward dynamic programming recursions (2.5).

Lemma 2.3: The functions $V_T^t(S)$ as given by (2.5) are

i) non-decreasing,

ii) concave.

Proof: The proofs are carried by backward induction. It is certainly true for V_T^{T+1} because β^{T+1} is non-decreasing and concave by assumption.

Suppose V_T^{t+1} is non-decreasing in S and let $S_1 \ge S_2$, then

$$V_T^t(S_1) = \max_{0 \le r \le S_1} \{\beta^t(r) - g^t(R - S_1, r) + \alpha V_T^{t+1}(S_1 - r)\} \ge$$

$$\max_{0 \le r \le S_2} \{\beta^t(r) - g^t(R - S_1, r) + \alpha V_T^{t+1}(S_1 - r)\} \ge$$

$$\max_{0 \le r \le S_2} \{\beta^t(r) - g^t(R - S_2, r) + \alpha V_T^{t+1}(S_2 - r)\} = V_T^t(S_2)$$

where the last step follows from the fact that g^t is non-decreasing in its first argument and V_T^{t+1} is non-decreasing by assumption. This completes the proof of (i).

To prove concavity let r_1^t and r_2^t be optimal solutions corresponding to $V_T^t(S_1)$ and $V_T^t(S_2)$ respectively. Then for any $0 \le \omega \le 1$

$$0 \le \omega r_1^t + (1 - \omega) r_2^t \le \omega S_1 + (1 - \omega) S_2$$

and

$$V_T^t(\omega S_1 + (1 - \omega) S_2) \ge \beta^t(\omega r_1^t + (1 - \omega) r_2^t) -$$

$$g^t(R - \omega S_1 + (1 - \omega) S_2, \ \omega r_1^t + (1 - \omega) r_2^t) +$$

$$\alpha V_T^{t+1}(\omega S_1 + (1 - \omega) S_2 - \omega r_1^t + (1 - \omega) r_2^t) \ge$$

$$\omega V_T^t(S_1) + (1 - \omega) V_T^t(S_2)$$

where the last step follows from the concavity of β^t and V_T^{t+1} and the convexity of g^t. This completes the induction. ||

<u>Lemma 2.4</u>: If the functions β^t, t=1,2,...,T+1, and e are piecewise linear with a finite number of segments then the functions $V_T^t(S)$ as given by (2.5) are piecewise linear with a finite number of segments.

Proof: The proof is carried by backward induction. Since $V_T^{T+1}(S) = \beta^{T+1}(S)$ the function V_T^{T+1} is piecewise linear with a finite number of segments by assumption. Suppose V_T^{t+1} is piecewise linear with a finite number of segments and let it be represented by

$$V_T^{t+1}(S) = \underset{m=1,\dots,M}{\text{minimum}} \{\nu^{t+1,m} + \zeta^{t+1,m}S\}$$

The function V_T^t can then be rewritten as

$$V_T^t(S) = -e(R - S) + \text{maximum}\ \{z^t + y^t + \alpha w^{t+1}\} \tag{2.24}$$

$$\begin{aligned} \text{s.t.}\quad z^t &\le b^{t,k} + \pi^{t,k}r && k=1,2,\dots,K^t \\ y^t &\le -e^j - \gamma^j(R - S + r) && j=1,2,\dots,J \\ w^{t+1} &\le \nu^{t+1,m} + \zeta^{t+1,m}S && m=1,2,\dots,M \\ 0 &\le r \le S && \end{aligned}$$

As shown in Lemma 2.1, the right-hand maximization results in a piecewise linear function of S with a finite number of segments. Since $V_T^t(S)$ is the difference of two piecewise linear functions with a finite number of segments, the lemma is proved.||

Conveniently enough $V_T^t(S)$ can be generated by parametric linear programming on (2.24).

The iterative approach described in section 2.III would imply solving at each iteration an approximated supplier's problem that can be seen as a relaxation of (2.23). The LP-supplier's problem that would be solved at each iteration has fewer constraints than (2.23) as $L^t \leq K^t$, for t=1,2,...,T. Each iteration either finds an equilibrium solution or generates another constraint to the LP-supplier's problem (2.23).

Obviously the supplier's problem can also be solved at each iteration by dynamic programming. For the present case this would involve computing at each iteration an approximation to the functions $\tilde{V}_T^t$, consisting of a relaxation of (2.24), when only a subset of K^t and M is considered. Apparently a more efficient procedure could result from the application of state-reduction algorithms such as the one proposed by Wilde [92]. By restricting the state-space (S) search, we can eliminate the expensive parametric linear programming over the entire state-space. Also specific simpler algorithms may be derived to deal with the dynamic programming recursions (2.5). A greedy algorithm is suggested in section 2.VII to deal with stationary piecewise linear revenue and extraction cost functions.

2.V. Extensions of the Basic Model

In this section we present some straightforward extensions of the basic model discussed in sections 2.II, 2.III and 2.IV. These are accompanied by remarks and or conditions that make them applicable.

2.V.1. Alternative Possibilities for Equilibrium

While the definition of the revenue function and the equilibrium con-

dition (2.9) makes the exposition of the decomposition approach in section 2.III more clear in terms of the iterative approach between sector and supplier, there are sensible alternative definitions. The present conditions (2.8) and (2.9) do not imply any loss of generality as we shall show in this section.

Alternative behavioral assumptions to the sector's willingness to pay to the supplier the full amount of its cost-savings could have been imposed. For example the sector may be willing to pay to the supplier only a fraction of its cost-savings. Also the existence of external agents absorbing a fraction of the supplier's revenue would imply in a modification of (2.8) and (2.9).

The revenue function could be defined alternatively as

$$\hat{\beta}^t(r) \equiv \psi^t[\beta^t(r),r]$$

and the equilibrium condition (2.9) would have to be modified accordingly. We shall argue that as long as ψ^t is concave and non-decreasing in its first argument the iterative approach converges to an equilibrium solution because

Lemma 2.5: If ψ^t is concave and non-decreasing in its first argument then $\hat{\beta}^t$ is concave.

Proof: The function β^t is concave because of definition (2.8) and Lemmas 2.1 and 2.2. Then for any $0 \leq \omega \leq 1$

$$\beta^t(\omega r_1 + (1 - \omega)r_2) \geq \omega\beta^t(r_1) + (1 - \omega)\beta^t(r_2)$$

Now

$$\hat{\beta}^t(\omega r_1 + (1 - \omega)r_2) \geq \psi^t[\omega\beta^t(r_1) + (1 - \omega)\beta^t(r_2), \omega r_1 + (1 - \omega)r_2] \geq$$

$$\omega\hat{\beta}^t(r_1) + (1 - \omega)\hat{\beta}^t(r_2)$$

because ψ^t is non-decreasing in its first argument (first step) and concave (last step).||

Several institutional instruments fall into the category described by Lemma 2.5. Among those we can cite:

i) Price Regulation:

$$\hat{\beta}^t(r) = \psi^t[\beta^t(r),r] = \min \{\bar{p}^t r,\ \beta^t(r)\}$$

where $\bar{p}^t$ is an exogenously given upper bound on the unit price that can be paid to the supplier.

ii) Cost of Externalities:

$$\hat{\beta}^t(r) = \psi^t[\beta^t(r),r] = \beta^t(r) - q^t(r)$$

where q^t is convex and increasing reflecting, for example, the social costs of pollution.

iii) Tax Depletion Allowance

$$\hat{\beta}^t(r) = \psi^t[\beta^t(r),r] = \left[1 + \frac{\eta\tau}{1-\tau}\right]\beta^t(r)$$

where η is the percentage of depletion allowance and τ is the tax rate. For $0 \leq \eta < 1$ the before-tax revenue function $\hat{\beta}^t$ is higher that β^t.

Three institutional possibilities were explored to take into account institutional agents that may exist between the economic sector and the supplier. These have been summarized by the *institutional operator* ψ^t. The extension of the iterative approach to include those external factors is summarized in Figure 2.5

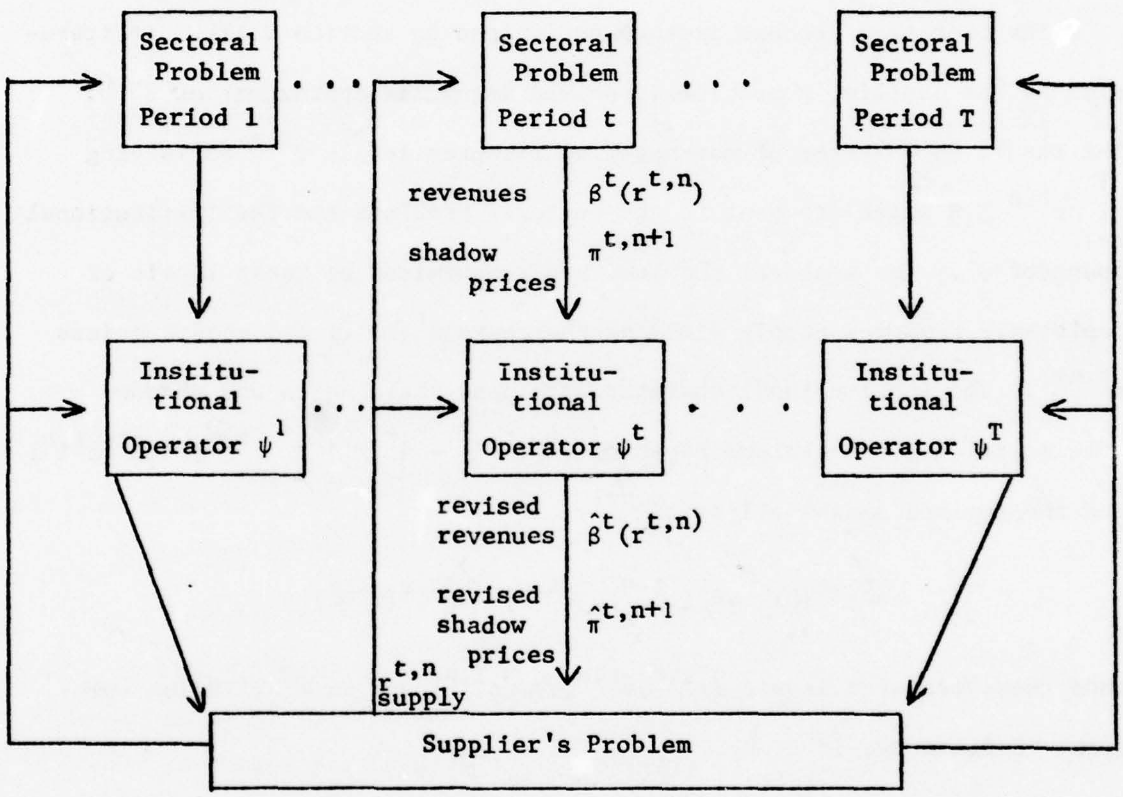


Figure 2.5

The iterative approach would involve solving the supplier's problem at each iteration using the new upper-bound approximations

$$\tilde{\hat{\beta}}^{t,n}(r) = \psi^t[\tilde{\beta}^{t,n}(r),r]$$

where $\tilde{\beta}^{t,n}$ is given by (2.13). Since we assumed ψ^t to be non-decreasing in its first argument $\tilde{\hat{\beta}}^{t,n}(r) \geq \hat{\beta}^t(r)$ for all $r \geq 0$. We shall concentrate for simplicity on examples of ψ^t yielding revenue functions $\hat{\beta}^t$ that are piecewise linear with a finite number of segments besides being concave.

The iterative process is then equivalent to section 2.III. At iteration n, the supplier's problem is solved using the approximation $\tilde{\hat{\beta}}^{t,n}$. The result is a vector of non-negative resource levels $r^{t,n}$ satisfying $\sum_{t=1}^{T} r^{t,n} \leq R$ which are sent to the sectoral problems and the institutional operator ψ^t. The sectoral problems are reoptimized at these levels of depletable resource supply yielding the costs $\Phi^t(r^{t,n})$ and shadow prices $\pi^{t,n+1}$. The institutional operator acts upon these costs and shadow prices yielding the revised revenues $\psi^t[\Phi^t(0) - \Phi^t(r^{t,n}), r^{t,n}] = \hat{\beta}^t(r^{t,n})$ and the revised shadow prices $\hat{\pi}^{t,n+1}$. If

$$\psi^t[\Phi^t(0) - \Phi^t(r^{t,n}), r^{t,n}] = \tilde{\hat{\beta}}^{t,n}(r^{t,n})$$

then these resource levels $(r^{1,n}, r^{2,n}, \ldots, r^{T,n})$ are an equilibrium solution. Conversely, if

$$\psi^t[\Phi^t(0) - \Phi^t(r^{t,n}), r^{t,n}] < \tilde{\hat{\beta}}^{t,n}(r^{t,n})$$

for some $t=1,2,\ldots,T$, a tighter approximation for the supplier's problem $\tilde{\hat{\beta}}^{t,n+1}$ is obtained by adding the new linear segment

$$\psi^t[\Phi^t(0) - \Phi^t(r^{t,n}), r^{t,n}] + \hat{\pi}^{t,n+1}(r - r^{t,n})$$

or equivalently

$$\hat{\beta}^t(r^{t,n}) + \hat{\pi}^{t,n+1}(r - r^{t,n}).$$

The arguments for infinite and finite convergence are similar to those leading to theorems 2.1 and 2.2. Convergence at a finite number of iterations would still follow from the finiteness of the number of segments that can be added to the approximations.

For the three institutional possibilities described above we have:

i) Price Regulation:

$$\hat{\beta}^t(r^{t,n}) = \min \{\bar{p}^t r^{t,n}, \Phi^t(0) - \Phi^t(r^{t,n})\}$$

and

$$\hat{\pi}^{t,n+1} = \begin{cases} \bar{p}^t & \text{if } \hat{\beta}^t(r^{t,n}) = \bar{p}^t r^{t,n} \\ \pi^{t,n+1} & \text{if } \hat{\beta}^t(r^{t,n}) = \Phi^t(0) - \Phi^t(r^{t,n}) \end{cases}$$

ii) Cost of Externalities

When in cost of externalities case q is given by a piecewise linear convex function:

$$q^t(r) = \underset{j=1,2,\ldots,J}{\text{maximum}} \{q_0^{t,j} + q_1^{t,j} r\}$$

then

$$\hat{\beta}^t(r^{t,n}) = \Phi^t(0) - \Phi^t(r^{t,n}) - q^t(r^{t,n})$$

and

$$\hat{\pi}^{t,n+1} = \pi^{t,n+1} - q_1^{t,j^*}$$

where j^* is the index that yields $q^t(r^{t,n})$, or equivalently q^{t,j^*} is the gradient of q^t at $r = r^{t,n}$.

iii) Tax Depletion Allowance

$$\hat{\beta}^t(r^{t,n}) = \left(1 + \frac{\eta\tau}{1-\tau}\right)[\Phi^t(0) - \Phi^t(r^{t,n})]$$

and

$$\hat{\pi}^{t,n+1} = \left[1 + \frac{\eta\tau}{1-\tau}\right]\pi^{t,n+1}$$

2.V.2. The Enriched Supplier's Problem

The supplier's problem as proposed in (2.1) has no other restrictions besides the depletable resource constraint. There are several possibilities to elaborate upon this formulation in order to make it more realistic. These are not necessarily mutually exclusive. However, the addition of other constraints to (2.1) is likely to make the supplier's problem too complex to be solved by dynamic programming methods for example. Other mathematical programming methods would be required. A few illustrative extensions are discussed in this context.

The supplier's problem can be extended to include extraction capacity constraints and investment decisions to enlarge those limits. If we let y^0 denote the initial extraction capacity available and y^t the amount of capacity installed in period t, we further require that

$$r^t \le \sum_{j=1}^{t} y^j + y^0 \qquad t=1,2,\dots,T \qquad (2.25)$$

Since capacity installment cost includes in general a fixed charge and a

variable component we need introduce the zero-one variables δ^t satisfying

$$\delta^t = \begin{cases} 0 & \text{if} \quad y^t = 0 \\ 1 & \text{if} \quad y^t > 0 \end{cases} \qquad t=1,2,\ldots,T \tag{2.26}$$

The present value of capacity build-up expenditures is then given by

$$\sum_{t=1}^{T} \alpha^{t-1} \{Q^t\delta^t + q^t y^t\} \tag{2.27}$$

where Q^t is the fixed charge and q^t is the cost per unit of capacity installed in period t.

An enriched supplier's problem would result from the simultaneous incorporation of (2.25) and (2.26) as constraints and (2.27) in the objective function of (2.1). The addition of the integer zero-one variables, however, further complicates the solution of the supplier's problem. The state-space for dynamic programming recursions is two dimensional; namely, the pair (Y,S) where Y is the total capacity available and S, as previously defined, is the stock of the depletable resource by the start of period t.

The supplier's problem would be even more complex if we were to take into account some of the intricacies of the capacity build-up process such as the lags existing in capacity expansion. While we have enough flexibility in the specification of the extraction cost-functions in (2.1), by allowing it to vary over time, it is more plausible to assume that transition to cheaper extraction technology can only be made at a fixed charge. The supplier's problem can be extended to include the decision on a tran-

sition period again by the addition of zero-one variables. We cannot neglect also the possibility of investments for depletable resource reserves expansion. However, since the outcome of a reserve expansion project is uncertain as the size of the new discoveries are not known a priori, this would lead us away from deterministic mathematical programming methods. Stochastic considerations are delayed until Chapter 4.

2.V.3. The Existence of Sectoral Minimum Resource Requirements

So far we have for the purpose of defining the revenue function (2.8) and a sectoral-supplier equilibrium (2.9) assumed that $\Phi^t(0) < +\infty$. However, it is likely that

$$\Phi^t(r) \begin{cases} = +\infty & 0 \le r < r^t_{min} \\ < +\infty & r^t_{min} \le r \end{cases} \tag{2.29}$$

in some time periods; or in other words, the economic sector has a minimum positive depletable resource requirement for feasible operations. Figures 2.6 and 2.7 illustrate the case of essential resources:

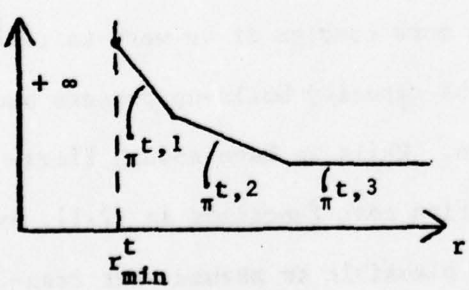


Figure 2.6

Figure 2.7

The situation (2.29) is depicted in Fig. 2.6. This type of cost function is more likely to arise in LP-sectoral problems because of the patterns of

substitution in linear programs. Fig. 2.7 shows a more smooth cost function for an essential resource that would result from a more general convex program.

Since $\Phi^t(r) \geq 0$, a necessary and sufficient condition for $\Phi^t(0) < +\infty$ is the existence of a feasible solution to the sectoral problems with $r = 0$. One possible method to check whether $\Phi^t(0) < +\infty$ is to solve in each period t the *minimum resource requirement problem*,

$$\begin{aligned} r^t_{min} &= \min \rho^t x^t \\ \text{s.t. } A^t_1 x^t - s^t &\leq 0 \\ A^t_2 x^t &\geq d^t \\ 0 \leq x^t,\ 0 \leq s^t &\leq s^t \end{aligned} \tag{2.30}$$

illustrated here for the case of LP-sectoral problems. If $r^t_{min} = 0$ the depletable resource is inessential to the sector's operations in period t. Otherwise it is essential. When $r^t_{min} = 0$, the optimal solution to (2.30) is feasible for (2.10) with $r = 0$ and consequently

$$\Phi^t(0) < +\infty.$$

It is conceivable that the situation described in (2.29) is more likely to arise in closed models such as the sectoral problems illustrated in section 2.III, since we have precluded the existence of perfect resource substitutes. If the economic sector has access to a perfect substitute of the depletable resource (e.g. imports) at a price p^t per unit, the sectoral problems need be revised. For example, the LP-sectoral problem

(2.10) is revised into:

$$\Phi'^{t}(r) = \min c^{t}x^{t} + f^{t}s^{t} + p^{t}m^{t} \tag{2.31a}$$

$$\text{s.t.} \quad \rho^{t}x^{t} \le r + m^{t} \tag{2.31b}$$

(2.10c), (2.10d) and (2.10e).

If $\Phi'^{t}(r)$ is finite for all $r \ge 0$, we have then equivalently to (2.11) and (2.12)

$$\Phi'^{t}(0) - \Phi'^{t}(r) =$$

$$\min_{k=1,2,\ldots,K^{t}} \{(\Phi'^{t}(0) - u'^{t,k}d^{t} + w'^{t,k}\, s^{t}) + \pi'^{t,k}r\} \tag{2.32}$$

where k indexes the set of extreme points of the dual problem

$$\begin{array}{llll} \max - \pi'^{t}r & & - w'^{t}\, s^{t} + u'^{t}d^{t} & \\ - \pi'^{t}\rho^{t} - v'^{t}A_1^{t} & & + u'^{t}A_2^{t} \le c^{t} & \\ & v'^{t} - w'^{t} & \le f^{t} & \quad (2.33) \\ \pi'^{t} & & \le p^{t} & \\ \pi'^{t} \ge 0,\ v'^{t} \ge 0,\ w'^{t} \ge 0,\ u'^{t} \ge 0. & & & \end{array}$$

We can then show that if the situation in (2.29) is overcome by the inclusion of a perfect resource substitute, the prices p^{t} will be upper-bounds on the prices the economic sector would be willing to pay the supplier. In the case of imports, import prices p^{t}, will limit the price per unit received by the domestic supplier.

<u>Lemma 2.6</u>: If $\Phi'^t(r)$ given by (2.31) is finite for all $r \geq 0$ and $\Phi^t(0) = +\infty$ in (2.10), then $\pi'^{t,1} = p^t$ (see Figure 3).

Proof: From the optimality conditions to (2.31) we have:

$$\pi'^{t,1} - p^t \leq 0$$

$$(\pi'^{t,1} - p^t)\hat{m}^t = 0.$$

Now we must have $\hat{m}^t > 0$ because otherwise $\hat{m}^t = 0$ would contradict that $\Phi^t(0) = +\infty$. Consequently,

$$\pi'^{t,1} = p^t. ||$$

While the inclusion of imports is likely to solve the economic sector's infeasibility problem by reducing the minimum domestic depletable resource requirements, it is still possible that the sector be non-operational at $r = 0$. This would be the case when trade barriers would limit the amount of resource that can be imported.

The existence of a minimum depletable resource requirement complicates the definition of a sensible revenue function. For $r^t < r^t_{min}$ the sector would be willing to pay nothing because it is still non-operational. In order to induce the suppliers to meet the minimum requirements the sector would have to decide on an arbitrarily large premium Π to pay for supplies satisfying $r^t \geq r^t_{min}$. We propose a redefinition of the revenue function (2.8) as:

$$\beta^t(r) = \begin{cases} 0 & 0 < r < r^t_{min} \\ \Pi^t & r = r^t_{min} \\ \Pi^t + \Phi^t(r^t_{min}) - \Phi^t(r) & r > r^t_{min} \end{cases} \qquad (2.34)$$

The shape of (2.34) is depicted in Figure 2.8.

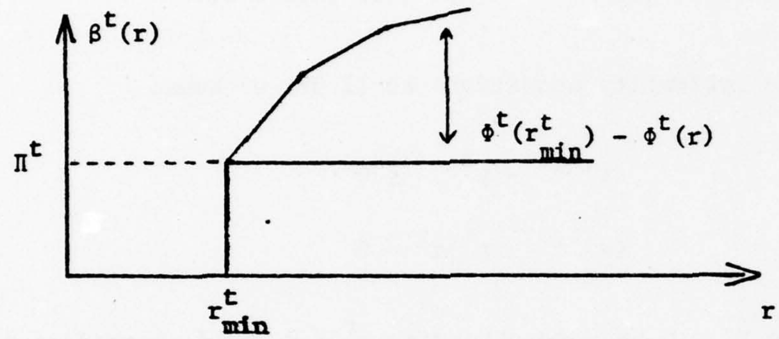


Figure 2.8

This redefinition of the revenue function complicates solving the supplier's problem (2.1) because of the fixed premium at $r = r^t_{min}$. This can be accomplished by the introduction of zero-one integer variables that would turn the supplier's problem into a mixed-integer program.

We have neglected in the above discussion the feasibility problems that may arise in this hypothetical economy due to sectoral minimum resource requirements. We require that $R \geq \sum_{t=1}^{T} r^t_{min}$, or in other words, that the reserves of depletable resource be large enough to satisfy the total sectoral minimum resource requirements over the planning horizon T.

As a final word, we should expect that the sectoral infeasibility in the absence of the depletable resource be overcome as long-run technologies substituting for the depletable resource utilization are implemented. The most likely situation is then one where $r^t_{min} > 0$ for $t < T_1$ and $\Phi^t(0) < +\infty$ for $t \geq T_1$.

2.VI. Relation to Previous Models

The supplier's problem (2.1) can be viewed as a discrete-time finite-horizon version of Gordon's [28] model, described in section 1.II.1, with:

$$\Pi^t(S^0 - S^t, r^t) = \beta^t(r^t) - g^t(S^0 - S^t, r^t)$$

and $S^0 = R$. However, our equilibrium approach turns it into a data-oriented model as the revenue functions β^t become endogenous, being generated by a sectoral model (see section 1.II.3.). Of course our equilibrium approach can be extended to infinitely long planning horizons. However, two major difficulties arise: 1) we would not want to solve at each iteration an infinite number of the sectoral problems (2.10) or (2.16); 2) in a non-stationary setting, the supplier's problem would be non-trivial to solve. While further considerations on infinite planning horizons are delayed until Chapter 5, some insight into the finite-horizon supplier's problem is gained by exploring some qualitative results of an infinite-horizon version of (2.1). Some of the conclusions can be related to previous results in resource constrained growth theory (see section 1.II.2.).

We formulate the *infinite-horizon supplier's problem* as

$$V_\infty(R) = \max \sum_{t=1}^{\infty} \alpha^{t-1}\{\beta^t(r^t) - g^t(\sum_{j=1}^{t-1} r^j, r^t)\}$$

$$\text{s.t.} \sum_{t=1}^{\infty} r^t \le R \qquad (2.35)$$

$$r^t \ge 0 \qquad t=1,2,\ldots.$$

For differentiable revenue functions β^t and extraction cost function g^t, the Kuhn-Tucker conditions for problem (2.35) require the existence of a pair $(\hat{r},\hat{\lambda})$ such that

$$\frac{d\beta^t}{dr^t} - \frac{dg^t}{dr^t} - \sum_{i=t+1}^{\infty} \alpha^{i-t} \frac{dg^i}{dr^t} - \hat{\lambda}\alpha^{1-t} \leq 0 \qquad t=1,2,\ldots. \qquad (2.36a)$$

$$\left(\frac{d\beta^t}{dr^t} - \frac{dg^t}{dr^t} - \sum_{i=t+1}^{T} \alpha^{i-t} \frac{dg^i}{dr^t} - \hat{\lambda}\alpha^{1-t}\right) \hat{r}^t = 0 \qquad t=1,2,\ldots. \qquad (2.36b)$$

$$\sum_{t=1}^{\infty} \hat{r}^t \leq R \qquad \hat{r}^t \geq 0 \qquad t=1,2,\ldots. \qquad (2.36c)$$

$$\hat{\lambda}\left(\sum_{t=1}^{\infty} \hat{r}^t - R\right) = 0 \qquad (2.36d)$$

$$\hat{\lambda} \geq 0 \qquad (2.36e)$$

Similarly to the optimality conditions (2.2) when the functions β^t and g^t are non-differentiable, (2.36) have to be restated in terms of linear combinations of the subgradients.

It is trivial to show that the functions $V_\infty^t(S)$ defined in (2.4) satisfy the recursions

$$V_\infty^t(S) = \underset{0\leq r\leq S}{\text{maximum}} \{\beta^t(r) - g^t(R - S,r) + \alpha V_\infty^{t+1}(S - r)\}.$$

The infinite-horizon supplier's problem is solved by computing

$$V_\infty(R) \equiv V_\infty^1(R).$$

Observations about the infinite-horizon optimal supply schedule can be carried over to the finite-horizon optimal supply schedule by an appro-

priate choice of the salvage function β^{T+1} in (2.1). The following lemma shows how a finite-horizon optimal supply schedule can be obtained as a truncation of the infinite-horizon optimal supply schedule.

<u>Lemma 2.7</u>: If $\beta^{T+1}(S) = V_{\infty}^{T+1}(S)$ then $r_T^t = r_{\infty}^t$, for $t=1,2,\ldots,T$, where r_{∞}^t is an optimal solution to (2.35), is optimal for (2.1).

Proof: The proof of this lemma is easily carried by backward induction by showing first that

$$V_T^T(S) = \underset{0 \le r^T \le S}{\text{maximum}} \{\beta^T(r^T) - g^T(R - S, r^T) + \alpha V_{\infty}^{T+1}(S - r^T)\} = V_{\infty}^T(S)$$

for all S.

Carrying the procedure backwards we will have

$$V_T^t(S) = V_{\infty}^t(S) \qquad t=T-1,T-2,\ldots,1$$

and also $V_T(R) \equiv V_T^1(R) = V_{\infty}^1(R) \equiv V_{\infty}(R)$. Hence it follows that $r_T^t = r_{\infty}^t$, $t=1,2,\ldots,T$ is an optimal solution to (2.1).||

The salvage function β^{T+1} may play a crucial role in the solution of the finite-horizon supplier's problem (2.1). If the disruptive effect of an arbitrarily selected salvage function is avoided by selecting $\beta^{T+1} = V_{\infty}^{T+1}$, the salvage function will play a major part only if in the solution of the infinite-horizon supplier's problem (2.35) $r_{\infty}^t > 0$ for some $t > T$. For this purpose the following result proves useful.

Lemma 2.8: If $\frac{\partial g^t}{\partial E}$ and $\frac{\partial g^t}{\partial r}$ are bounded for $E \geq 0$, $r \geq 0$, $t=1,2,\ldots,$ and $r_\infty^{t_1} > 0$ for some period t_1 then $r_\infty^t > 0$ for all periods in which

$$\lim_{r \to 0^+} \frac{d\beta^t}{dr} = +\infty.$$

Proof: Let $\hat{r}^{t_1} > 0$, then from (2.36b)

$$\hat{\lambda}\alpha^{1-t_1} = \frac{d\beta^{t_1}}{dr^{t_1}} - \frac{dg^{t_1}}{dr^{t_1}} - \sum_{i=t_1+1}^{\infty} \alpha^{i-t} \frac{dg^i}{dr^{t_1}}$$

which implies $\hat{\lambda} < +\infty$ since $\frac{dg^i}{dr^{t_1}}$ is bounded by assumption for $i=t_1, t_1 + 1, \ldots.$ Suppose there exists a time period $t_2 \neq t_1$ with $\lim_{r \to 0^+} \frac{d\beta^{t_2}}{dr} = +\infty$ such that $\hat{r}_\infty^{t_2} = 0$. Then we would have from (2.36) that $\hat{\lambda} = +\infty$ which is a contradiction. Hence we must have $\hat{r}^{t_2} > 0.||$

What the above lemma suggests is that in a truncation of the infinite-horizon supplier's problem under the conditions of Lemma 2.7, particular attention should be paid to those time periods for which the marginal revenue approaches infinite as resource levels approach zero. While this situation cannot happen if β^t is generated by a linear programming sectoral problem such as (2.10), because the piecewise linearity proved in Lemma 2.1 precludes this possibility, it is conceivable that more general convex sectoral problem formulations such as (2.16) may lead to revenue functions depicted below:

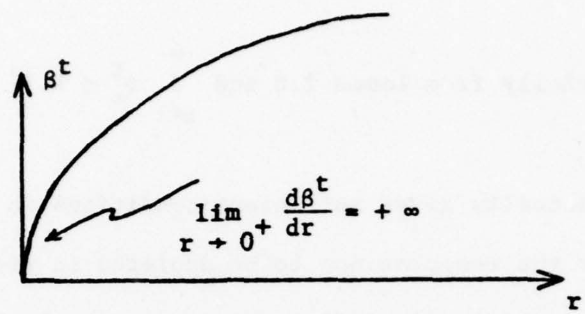


Figure 2.9

The stationary infinite-horizon supplier's problem with

$$\beta^t = \beta \qquad \text{for } t=1,2,\ldots.$$

$$g^t = g \qquad \text{for } t=1,2,\ldots.$$

is particularly important because given the difficulties in forecasting over long planning horizons it is usual practice to assume stationarity at least after a certain time period. In our equilibrium approach this would require assuming that the economic sector model is stationary; namely

$$\phi^t = \phi \qquad \text{for } t=1,2,\ldots.$$

The implications of Lemma 2.7 for a stationary infinite-horizon supplier's problem are given by:

<u>Corollary 2.1</u>: If $\frac{\partial g}{\partial E}$ and $\frac{\partial g}{\partial r}$ are bounded and $\lim_{r \to 0^+} \frac{d\beta}{dr} = +\infty$, then $\hat{r}_\infty^t > 0$ for $t=1,2,\ldots$ with $\lim_{t \to \infty} \hat{r}_\infty^t = 0$.

Proof: Follows trivially from Lemma 2.8 and $\sum_{t=1}^{\infty} r_{\infty}^{t} \leq R.$||

Corollary 2.1 actually gives sufficient conditions in terms of the marginal revenue for the resource not to be depleted in finite time. Similarly to DasGupta and Heal's [14] results (see section 1.II.2.) finite or infinite time depletion rests not only on the (in)essentiality of the depletable resource (in our case, $\Phi(0)$ (<) = $+\infty$) on the behavior of the economic sector's marginal cost-savings. The existing connection between the marginal sectoral cost-savings and the resource marginal productivity relates the two results.

From the optimality conditions (2.36) it is trivial to verify that for stationary problems discounting will be responsible for a non-increasing supply schedule over time paralleling DasGupta and Heal [14], Solow [81], Koopmans [52] and others. Also the effect of the discount rate seems to lead to the obvious conclusion that smaller discount rates will speed up depletion. Observation of (2.36a) and (2.36b) leads to Hotelling's [44], Gordon's [28] and Weinstein and Zeckhauser's [91] conclusions; namely that for extraction cost functions that are independent of past cumulative extraction, the optimal solution will imply that marginal revenue minus marginal extraction cost will increase at the discount rate. The numerical example in the next section will illustrate some of the above assertions.

2.VII. A Numerical Example

We shall now illustrate the application of the iterative approach of

section 2.III. to our equilibrium model. For the sectoral model we assume a stationary LP-sectoral problem (2.10). The structure of the LP-sectoral problem is inherited from BESOM [3].

This illustrative example counts ten end-use demands d_j, j=1,2,...,10 and four alternative primary supplies s_i, i=1,2,3,4 in addition to the depletable resource (i=5)[1]. The decision variables are the purchase of the alternative primary supplies s_i(i=1,2,3,4), the amount of depletable resource imports m and the activities x_{ij}, representing the flows in the links (i,j) with efficiency a_{ij}. The costs of utilizing devices in the links (i,j) are given by c_{ij} and f_i gives the prices of the alternative primary supplies. Imports cost p per unit. Under the assumption of stationarity over the planning horizon T, we have for t=1,2,...,T,

$$\Phi^t(r) = \Phi(r) = \min \sum_{i=1}^{5} \sum_{j=1}^{10} c_{ij}x_{ij} + \sum_{i=1}^{4} f_i s_i + pm$$

$$\text{s.t.} \quad \sum_{j=1}^{10} x_{5j} \qquad \leq r + m$$

$$\sum_{j=1}^{10} x_{ij} \qquad - s_i \leq 0$$

$$i=1,2,3,4$$

---

1 There has been an attempt to identify the primary supplies with nuclear fuels (1), hydropower (2), coal (3), natural gas (4) and crude oil (5). Also end-uses would correspond to automobiles (1), bus, process heat and miscellaneous industrial uses (4), petrochemicals (5), other industrial uses (6), space heat (7), miscellaneous thermal uses (8), miscellaneous electric uses (9), and air conditioning (10). However, due to the difficulties in obtaining and disaggregating BESOM's cost and efficient parameters into this simple 5 x 10 structure, the results derived do not intend to be conclusive but illustrative. Nevertheless, in the subsequent footnotes we shall identify the origin of the primary data utilized.

$$\sum_{i=1}^{5} a_{ij}x_{ij} \geq d_j \qquad j=1,2,\ldots,10$$

$$0 \leq x_{ij} \qquad i=1,2,\ldots,5 \qquad j=1,2,\ldots,10$$

$$0 \leq s_i \leq s_i \qquad i=1,2,3,4.$$

The data utilized are presented in Tables 2.1, 2.2, 2.3 and 2.4. Taking as the price of imports[1]

$$p = 11$$

and as the amount of resource reserves[2],

$$R = 213$$

we obtained the cost function $\Phi(r)$ by parametric linear programming as depicted in Fig. 2.10. The corresponding sequence of shadow prices π is given in Tab. 2.5. From Tab .2.5 and Fig.2.10, we can notice that $r \leq 27.36$ (the first breakpoint in the cost function Φ) the depletable resource is substituting for imports ($\pi^{t,1} = p$). Only for a supply exceeding 27.36 units, the resource will start substituting for the alternative primary supplies.

Before illustrating the iterative approach from section 2.III. to determine equilibrium in our basic model under different assumptions on the supplier's discount factor, planning horizon, extraction costs and salvage

---

[1] In compatible units a price of $\$11/10^6$ B.T.U. would correspond to \$63.80/bbl of crude oil. Although unrealistic, this choice led to a function that had enough linear pieces to make the example attractive.

[2] The amount of reserves is in compatible units 213×10^{15} B.T.U. which is the size of proved reserves in the U.S. used by Nordhaus [66].

Table 2.1. Data for Alternative Primary Supplies

Supply i	Upper Bounds[1] s_i	Prices[2] f_i
1	.58	.30
2	2.89	0
3	12.60	.60
4	23.12	1.82

[1]The data is given in 10^{15} B.T.U. and obtained from BESOM [3] 1972 data.

[2]Prices are measured in dollar per 10^6 B.T.U. The source is BESOM [3] 1985 cost data.

Table 2.2. Data for End-Use Demands

End-Use j	Demands[1] d_j
1	9.12
2	4.39
3	2.98
4	7.61
5	4.19
6	2.96
7	5.31
8	1.93
9	1.89
10	1.03

[1]The demand data is given in 10^{15} B.T.U. The source is BESOM [3] 1972 demand data.

Table 2.3. Cost Data[1]

End-Use j / Supply i	1	2	3	4	5	6	7	8	9	10
1				11.81		11.81	2.0	7.50	10.24	14.25
2				5.84		5.84	5.84	4.15	5.84	5.84
3				12.07	.77	.94	3.51	.9	8.31	14.22
4				1.48	.96	1.48	1.62	3.62	9.99	12.60
5	3.04	2.56	2.21	2.29	1.77	1.89	2.56	1.56	12.20	2.29

[1]Source is BESOM [3] 1985 cost data and is given in dollar per 10^6 B.T.U.

Table 2.4. Efficiency Data[1]

End-Use j / Supply i	1	2	3	4	5	6	7	8	9	10
1				.279		.273	.279	.279	.279	.767
2				.297		.291	.297	.297	.297	.817
3				.679	.970	.270	.380	.285	.285	.780
4				.582	.910	.261	.437	.482	.267	1.64
5	1	1	1	.680	1	.293	.432	.385	.293	.807

[1]Efficiency data is taken from BESOM [3] 1972 data. Because we assume no intermediate form of energy weighted averages were taken when multiple paths between supply i and end-use j existed in the original BESOM.

Table 2.5. Shadow Prices

Range of r		Shadow Prices	
minimum	maximum		
0	27.36	$\pi^{t,1}$	11
27.36	27.41	$\pi^{t,2}$	7.93
27.41	28.15	$\pi^{t,3}$	7.16
28.15	29.74	$\pi^{t,4}$	5.42
29.74	30.26	$\pi^{t,5}$	1.80
30.26	35.71	$\pi^{t,6}$	.40
35.71	39.69	$\pi^{t,7}$	.35
39.69	∞	$\pi^{t,8}$	0

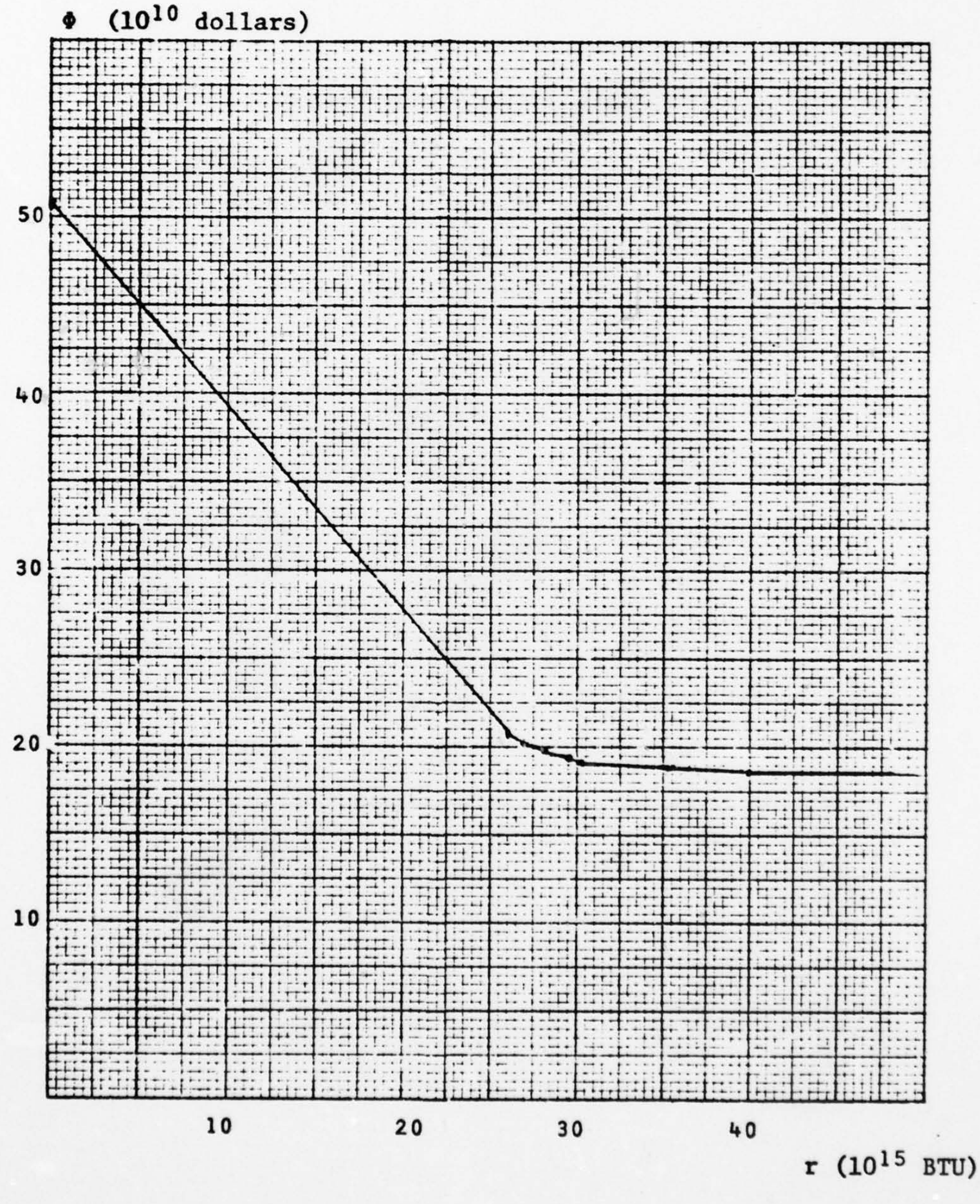


Figure 2.10. Cost of Meeting Demand for End-Use Goods as a Function of Domestic Supply.

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NORMATIVE MODELS OF DEPLETABLE RESOURCES.(U)
MAY 78 E M MODIANO DAAG29-76-C-0064

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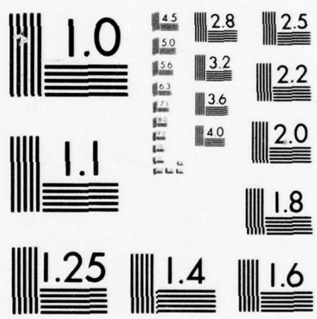

MICROCOPY RESOLUTION TEST CHART
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function, we show the solution to a related problem that will lead to a greedy algorithm to solve the supplier's problem with constant marginal extraction cost. Its extension to cover the case of piecewise linear cumulative extraction cost functions is discussed afterwards.

Consider the linear programming problem

$$\max \ cx \tag{2.38a}$$

$$\text{s.t. } ex = b \tag{2.38b}$$

$$0 \le x \le \bar{x} \tag{2.38c}$$

where $c \ge 0$, $b \ge 0$ and e is a row vector of ones. Letting λ and θ be the dual variables associated with constraints (2.38b) and (2.38c) respectively, the optimality conditions for problem (2.38) are

$$c_i - \lambda - \theta_i \le 0$$

$$(c_i - \lambda - \theta_i)x_i = 0$$

$$ex = b$$

$$0 \le x \le \bar{x}$$

$$(x_i - \bar{x}_i)\theta_i = 0$$

$$0 \le \theta_i.$$

By assuming without loss of generality that $c_i \ge c_{i+1}$, it is easy to show that the solution to (2.38) is given recursively by

$$x_i = \min \{b - \sum_{j=1}^{i-1} x_j, \bar{x}_i\}$$

with

$$\lambda = \begin{cases} c_I & \text{if} \quad x_I < \bar{x}_I \quad \text{and} \quad x_i = 0 \text{ for } i > I \\ c_{I+1} & \text{if} \quad x_I = \bar{x}_I \quad \text{and} \quad x_i = 0 \text{ for } i > I \end{cases}$$

$$\theta_i = \min \{c_i - \lambda, 0\}$$

We may think of problem (2.38) in terms of determining the best allocation of b units of a resource to blocks with maximum capacity $\bar{x}_i$ and payoff c_i per unit. The greedy algorithm will find an optimal solution to (2.38) by filling those blocks in descending order of the marginal profits c_i.

For constant marginal extraction cost functions

$$g(\sum_{j=1}^{t-1} r^j, r^t) = gr^t$$

and piecewise linear revenue and salvage functions, the supplier's problem objective will be piecewise linear and concave as depicted below:

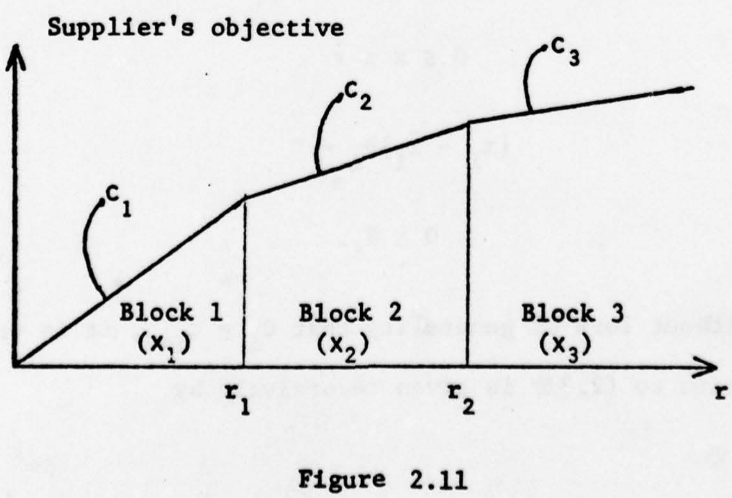


Figure 2.11

Dantzig [11] showed how this objective is rewritten in terms of allocation blocks. This is also depicted in Figure 9. In our case it will require merging and ordering descendingly the sequences

$$\{\alpha^{t-1}[\pi^{t,k} - g]\} \quad \text{and} \quad \{\alpha^{T}\pi^{T+1,k}\}.$$

Identifying the resulting sequence with the sequence $\{c_i\}$ and letting $\bar{x}_i = r_i - r_{i-1}$, the supplier's problem can be written in the format (2.38). The application of this greedy algorithm is illustrated in the following examples 1.1 through 1.5 (Tables 2.6 - 2.10) only for the last iteration of the iterative scheme. The numbers in upper case in those tables indicate the allocation order.

When the extraction cost functions are specified by a piecewise linear convex cumulative extraction cost function (2.20) the application

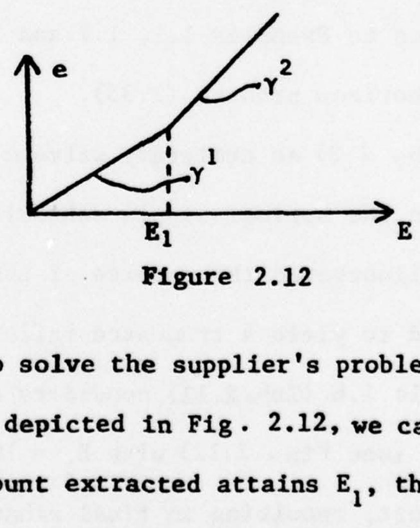


Figure 2.12

of greedy algorithms to solve the supplier's problem becomes more complex. For the two-piece case depicted in Fig. 2.12, we can readily observe that once the cumulative amount extracted attains E_1, the greedy algorithm with $g = \gamma^2$ can be utilized to determine the optimal allocation over time of the remaining $R - E_1$ units. The major difficulty stems, however, from determining the transition period. While we claim that in this simple case an optimal solution may be found by means of an efficient trial-and-error

search procedure, we recognize that piecewise linear cumulative extraction cost functions with more than two linear pieces might not be efficiently handled by greedy algorithms.

Example 1.1 (Tab.2.6) considers the case of zero extraction costs and zero salvage value. This case has been studied by Weinstein and Zeckhauser [91] and to some extent by Hotelling [44], and our results confirm their assertions as sequence of optimal marginal revenues

$$\{5.27, 5.85, 6.50, 7.22, 8.02, 8.91, 9.9, 11\}$$

is increasing geometrically over time at the interest rate $(1/\alpha)$. The sequence of present value marginal revenues is constant over time (λ=5.27). Example 1.2 (Tab.2.7) introduces constant marginal extraction costs and Example 1.3 (Tab.2.8) considers the effect of a smaller discount factor in increasing the rate of extraction in the early periods. It is trivial to observe that the solutions to Examples 1.1, 1.2 and 1.3 are solutions to the stationary infinite-horizon problem (2.35).

For Example 1.4 (Tab. 2.9) an arbitrary salvage function was selected. Its disruptive effect in the optimal supply schedule is readily noticed. Example 1.5 (Tab.2.10) illustrates the results of Lemma 2.7, as the salvage function is selected to yield a truncated infinite-horizon supply schedule. Lastly, Example 1.6 (Tab. 2.11) considers a two-piece cumulative extraction cost function (see Fig. 2.12) with $E_1 = 100$, $\gamma^1 = 1$ and $\gamma^2 = 2$. All six examples are robust, resulting in final exhaustion of the depletable resource reserves at the end of the eighth period.

Table 2.6 - Example 1.1: $\alpha = .9$, $T = 10$, $g^t(\sum_{j=1}^{t-1} r^j, r^t) = 0$ and $\beta^{11}(S) = 0$.

Iteration n	Known Shadow Prices $L^{t,n}$	Supply Schedule $r^{1,n}$	$r^{2,n}$	$r^{3,n}$	$r^{4,n}$	$r^{5,n}$	$r^{6,n}$	$r^{7,n}$	$r^{8,n}$	$r^{9,n}$	$r^{10,n}$
1	$\pi^{t,1}$	213	0	0	0	0	0	0	0	0	0
2	$\pi^{t,1},\pi^{t,8}$	29.07	29.07	29.07	29.07	29.07	29.07	29.07	9.51	0	0
3	$\pi^{t,1},\pi^{t,4},\pi^{t,8}$	30.57	27.57	27.57	27.57	27.57	27.57	27.57	17.01	0	0
4	$\pi^{t,1},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	30.57	28.15	28.15	27.37	27.37	27.37	27.37	16.65	0	0
5	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	30.57	28.15	28.15	27.41	27.36	27.36	27.36	16.64	0	0
6	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,6},\pi^{t,8}$	29.87	28.15	28.15	27.41	27.36	27.36	27.36	17.34	0	0
7*	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,5},\pi^{t,6},\pi^{t,8}$	29.74	28.15	28.15	27.41	27.36	27.36	27.36	17.47	0	0

*At iteration 7 the solution is optimal with: $V_{10}(213) = 1{,}686.36$

Table 2.6 (continued) - Iteration 7

Discounted Marginal Profit	Period t 1	2	3	4	5	6	7	8	9	10
$\alpha^{t-1}[\pi^{t,1}]$	11.0^{1}	9.90^{2}	8.91^{3}	8.02^{4}	7.22^{6}	6.50^{9}	5.85^{12}	5.27^{16}	4.74	4.27
$\alpha^{t-1}[\pi^{t,2}]$	7.93^{5}	7.14^{8}	6.43^{11}	5.79^{14}	5.21	4.69	4.22	3.80	3.42	3.08
$\alpha^{t-1}[\pi^{t,3}]$	7.16^{7}	6.44^{10}	5.80^{13}	5.22	4.70	4.23	3.81	3.43	3.09	2.78
$\alpha^{t-1}[\pi^{t,4}]$	5.42^{15}	4.88	4.39	3.95	3.56	3.20	2.88	2.59	2.33	2.10
$\alpha^{t-1}[\pi^{t,5}]$	1.80	1.62	1.46	1.31	1.18	1.06	.95	.86	.77	.69
$\alpha^{t-1}[\pi^{t,6}]$	.40	.36	.32	.29	.26	.23	.21	.19	.17	.15
$\alpha^{t-1}[\pi^{t,8}]$	0	0	0	0	0	0	0	0	0	0

Table 2.7 - Example 1.2: $\alpha = .9$, $T = 10$, $g^t(\sum_{j=1}^{t-1} r^j, r^t) = r^t$, and $\beta^{11}(S) = 0$

Iteration n	Known Shadow Prices $L^{t,n}$	Supply Schedule $r^{1,n}$	$r^{2,n}$	$r^{3,n}$	$r^{4,n}$	$r^{5,n}$	$r^{6,n}$	$r^{7,n}$	$r^{8,n}$	$r^{9,n}$	$r^{10,n}$
1	$\pi^{t,1}$	213	0	0	0	0	0	0	0	0	0
2	$\pi^{t,1}, \pi^{t,8}$	29.07	29.07	29.07	29.07	29.07	29.07	29.07	9.51	0	0
3	$\pi^{t,1}, \pi^{t,4}, \pi^{t,8}$	27.57	27.57	27.57	27.57	27.57	27.57	27.57	20.01	0	0
4	$\pi^{t,1}, \pi^{t,3}, \pi^{t,4}, \pi^{t,8}$	28.15	28.15	28.15	27.37	27.37	27.37	27.37	19.07	0	0
5*	$\pi^{t,1}, \pi^{t,2}, \pi^{t,3}, \pi^{t,4}, \pi^{t,8}$	28.15	28.15	28.15	27.41	27.36	27.36	27.36	19.06	0	0

*At iteration 5 the solution is optimal with: $V_{10}(213) = 1,532.15$

Table 2.7 (continued) - Iteration 5

Discounted Marginal Profit	Period t: 1	2	3	4	5	6	7	8	9	10
$\alpha^{t-1}[\pi^{t,1}-1]$	10.0^{1}	9.0^{2}	8.10^{3}	7.29^{4}	6.56^{6}	5.90^{9}	5.31^{12}	4.78^{15}	4.30	3.87
$\alpha^{t-1}[\pi^{t,2}-1]$	6.93^{5}	6.24^{7}	5.61^{10}	5.05^{13}	4.55	4.09	3.68	3.31	2.98	2.68
$\alpha^{t-1}[\pi^{t,3}-1]$	6.16^{8}	5.54^{11}	4.99^{14}	4.49	4.04	3.64	3.27	2.95	2.65	2.39
$\alpha^{t-1}[\pi^{t,4}-1]$	4.42	3.98	3.58	3.22	2.90	2.61	2.35	2.11	1.90	1.71
$\alpha^{t-1}[\pi^{t,8}-1]$	-1.0	-.90	-.81	-.73	-.66	-.59	-.53	-.48	-.43	-.39

Table 2.8 - Example 1.3: $\alpha = .7$, $T = 10$, $g^t(\sum_{j=1}^{t-1} r^j, r^t) = r^t$ and $\beta^{11}(S) = 0$

Iteration n	Known Shadow Prices $L^{t,n}$	Supply Schedule $r^{1,n}$	$r^{2,n}$	$r^{3,n}$	$r^{4,n}$	$r^{5,n}$	$r^{6,n}$	$r^{7,n}$	$r^{8,n}$	$r^{9,n}$	$r^{10,n}$
1	$\pi^{t,1}$	213	0	0	0	0	0	0	0	0	0
2	$\pi^{t,1},\pi^{t,8}$	29.07	29.07	29.07	29.07	29.07	29.07	29.07	9.51	0	0
3	$\pi^{t,1},\pi^{t,4},\pi^{t,8}$	30.57	30.57	30.57	30.57	30.57	27.57	27.57	5.01	0	0
4	$\pi^{t,1},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	30.57	30.57	30.57	30.57	30.57	28.15	27.37	4.63	0	0
5	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	30.57	30.57	30.57	30.57	30.57	28.15	27.36	4.64	0	0
6	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,6},\pi^{t,8}$	29.87	29.87	29.87	29.87	29.87	28.15	27.36	8.14	0	0
7*	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,5},\pi^{t,6},\pi^{t,8}$	29.74	29.74	29.74	29.74	29.74	28.15	27.36	8.79	0	0

*At iteration 7 the solution is optimal with $V_{10}(213) = 878.04$

Table 2.8 (continued) - Iteration 7

Discounted Marginal Profit	Period t: 1	2	3	4	5	6	7	8	9	10
$\alpha^{t-1}[\pi^{t,1}-1]$	10.0^{1}	7.0^{2}	4.90^{5}	3.43^{9}	2.40^{13}	1.68^{17}	1.18^{21}	$.82^{25}$	.58	.41
$\alpha^{t-1}[\pi^{t,2}-1]$	6.93^{3}	4.85^{6}	3.40^{10}	2.38^{14}	1.66^{18}	1.16^{22}	.82	.57	.40	.28
$\alpha^{t-1}[\pi^{t,3}-1]$	6.16^{4}	4.31^{8}	3.02^{12}	2.11^{16}	1.48^{20}	1.04^{24}	.72	.51	.36	.25
$\alpha^{t-1}[\pi^{t,4}-1]$	4.42^{7}	3.09^{11}	2.17^{15}	1.52^{19}	1.06^{23}	.74	.52	.36	.25	.18
$\alpha^{t-1}[\pi^{t,5}-1]$	.80	.56	.39	.27	.19	.13	.09	.07	.05	.04
$\alpha^{t-1}[\pi^{t,6}-1]$	-.60	-.42	-.29	-.21	-.14	-.10	-.07	-.05	-.03	-.02
$\alpha^{t-1}[\pi^{t,8}-1]$	-1.0	-.70	-.49	-.34	-.24	-.17	-.12	-.08	-.06	-.04

Table 2.9 - Example 1.4: $\alpha = .9$, $T = 5$, $g^t(\sum_{j=1}^{t-1} r^j, r^t) = r^t$ and $\beta^6(S) = 10.16S$

Itera-tion n	Known Shadow Prices $L^{t,n}$	Supply Schedule $r^{1,n}$	$r^{2,n}$	$r^{3,n}$	$r^{4,n}$	$r^{5,n}$	Stock Left Over $S^{6,n}$
1	$\pi^{t,1}$	213	0	0	0	0	0
2	$\pi^{t,1},\pi^{t,8}$	29.07	29.07	29.07	29.07	29.07	67.65
3	$\pi^{t,1},\pi^{t,4},\pi^{t,8}$	27.57	27.57	27.57	27.57	27.57	75.15
4	$\pi^{t,1},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	28.15	27.37	27.37	27.37	27.37	75.37
5*	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	28.15	27.41	27.36	27.36	27.36	75.36

*At iteration 5 the solution is optimal with $V_5(213) = 1,577.83$

Table 2.9 (continued) - Iteration 5

Discounted Marginal Profit	Period t 1	2	3	4	5	6
$\alpha^{t-1}[\pi^{t,1}-1]$	10.0[1]	9.0[2]	8.10[3]	7.29[4]	6.56[6]	
$\alpha^{t-1}[\pi^{t,2}-1]$	6.93[5]	6.24[7]	5.61	5.05	4.55	
$\alpha^{t-1}[\pi^{t,3}-1]$	6.16[8]	5.54	4.99	4.49	4.04	
$\alpha^{t-1}[\pi^{t,4}-1]$	4.42	3.98	3.58	3.22	2.90	
$\alpha^{t-1}[\pi^{t,8}-1]$	-1.0	-.90	-.81	-.73	-.66	
$\alpha^{T}\pi^{T+1}$						6.0[9]

Table 2.10 - Example 1.5: $\alpha = .9$, $T = 5$, $g^t(\sum_{j=1}^{t-1} r^j, r^t) = r^t$ and $\beta^6(S) = V_\infty^6(S)$

Itera-tion	Known Shadow Prices	Supply Schedule					Stock Left Over
n	$L^{t,n}$	$r^{1,n}$	$r^{2,n}$	$r^{3,n}$	$r^{4,n}$	$r^{5,n}$	$r^{6,n}$
1	$\pi^{t,1}$	213	0	0	0	0	0
2	$\pi^{t,1},\pi^{t,8}$	29.07	29.07	29.07	29.07	29.07	67.65
3	$\pi^{t,1},\pi^{t,4},\pi^{t,8}$	27.57	27.57	27.57	27.57	27.57	75.15
4	$\pi^{t,1},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	28.15	28.15	28.15	27.37	27.37	73.81
5*	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	28.15	28.15	28.15	27.41	27.36	73.78

*At iteration 5 the solution is optimal with $V_5(213) = 1{,}532.15$

Table 2.10 (continued) - Iteration 5

Discounted Marginal Profit	Period t 1	2	3	4	5	6
$\alpha^{t-1}[\pi^{t,1}-1]$	10.0^{1}	9.0^{2}	8.10^{3}	7.29^{4}	6.56^{6}	
$\alpha^{t-1}[\pi^{t,2}-1]$	6.93^{5}	6.24^{7}	5.61^{10}	5.05^{13}	4.55	
$\alpha^{t-1}[\pi^{t,3}-1]$	6.16^{8}	5.54^{11}	4.99^{14}	4.49	4.04	
$\alpha^{t-1}[\pi^{t,4}-1]$	4.42	3.98	3.58	3.22	2.90	
$\alpha^{t-1}[\pi^{t,8}-1]$	-1.0	-.90	-.81	-.73	-.66	
$\alpha^{T}\pi^{T+1,1}$						5.90^{9}
$\alpha^{T}\pi^{T+1,2}$						5.31^{12}
$\alpha^{T}\pi^{T+1,3}$						4.78^{15}
$\alpha^{T}\pi^{T+1,4}$						4.09
⋮						⋮

Table 2.11- Example 1.6: $\alpha = .9$, $T = 10$, $g^t(E,r^t) = \begin{cases} r^t & E \le 100, r^t \le 100-E \\ 100-E+2(r^t-100+E) & E \le 100, r^t \ge 100-E,\ \beta^{11}(S)=0 \\ 2r^t & E \ge 100 \end{cases}$

Iteration n	Known Shadow Prices $L^{t,n}$	Supply Schedule $r^{1,n}$	$r^{2,n}$	$r^{3,n}$	$r^{4,n}$	$r^{5,n}$	$r^{6,n}$	$r^{7,n}$	$r^{8,n}$	$r^{9,n}$	$r^{10,n}$
1	$\pi^{t,1}$	213	0	0	0	0	0	0	0	0	0
2	$\pi^{t,1},\pi^{t,8}$	29.07	29.07	29.07	29.07	29.07	29.07	29.07	9.51	0	0
3	$\pi^{t,1},\pi^{t,4},\pi^{t,8}$	27.57	27.57	27.57	27.57	27.57	27.57	27.57	20.01	0	0
4	$\pi^{t,1},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	28.15	28.15	27.37	27.37	27.37	27.37	27.37	19.85	0	0
5*	$\pi^{t,1},\pi^{t,2},\pi^{t,3},\pi^{t,4},\pi^{t,8}$	28.15	28.15	27.41	27.41	27.36	27.36	27.36	19.80	0	0

*At iteration 5 the solution is optimal with: $V_{10}(213) = 1,465.76$

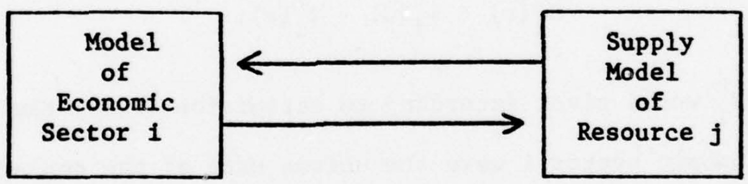
Table 2.11 (continued) - Iteration 5

Discounted Marginal Profit	Period t 1	2	3	4	5	6	7	8	9	10
$\alpha^{t-1}[\pi^{t,1}-1]$	$\boxed{10.0}$	$\boxed{9.0}$	$\boxed{8.10}$	$\boxed{7.29}$						
$\alpha^{t-1}[\pi^{t,1}-2]$				$\boxed{6.56}$	$\boxed{5.90}$	$\boxed{5.31}$	$\boxed{4.78}$	$\boxed{4.30}$	3.87	3.48
$\alpha^{t-1}[\pi^{t,2}-1]$	$\boxed{6.93}$	$\boxed{6.24}$	$\boxed{5.61}$	5.05						
$\alpha^{t-1}[\pi^{t,2}-2]$				$\boxed{4.32}$	3.89	3.50	3.15	2.84	2.55	2.30
$\alpha^{t-1}[\pi^{t,3}-1]$	$\boxed{6.16}$	$\boxed{5.54}$	4.99	4.49						
$\alpha^{t-1}[\pi^{t,3}-2]$				3.76	3.39	3.05	2.74	2.47	2.22	2.00
$\alpha^{t-1}[\pi^{t,4}-1]$	4.42	3.98	3.58	3.22						
$\alpha^{t-1}[\pi^{t,4}-2]$				2.49	2.24	2.02	1.82	1.64	1.47	1.32
$\alpha^{t-1}[\pi^{t,8}-1]$	-1.0	-.90	-.81	-.73						
$\alpha^{t-1}[\pi^{t,8}-2]$				-1.46	-1.31	1.18	1.06	.96	.86	.77

Chapter 3: Equilibrium Models with Multiple Economic Agents and Multiple Depletable Resources

3.I. Introduction

Chapter 2 introduced an equilibrium model between a unique economic sector and a unique supplier of a depletable resource. In the present chapter this analysis is extended to cover the general case of multiple economic sectors, multiple suppliers and multiple depletable resources. Our final goal is the integration of sectoral and supply models of depletable resources through an iterative scheme depicted below:



In the following three sections we will consider cases which essentially fall back into the framework of Chapter 2; these are: multiple economic sectors paying their cost-savings in meeting demand, multiple (collusive) suppliers and perfectly competitive markets. In the subsequent section the impact of some of the different market structures will be considered. The last section in this chapter considers the case of multiple depletable resources.

3.II. Multiple Economic Sectors

While we have assumed in Chapter 2 that the depletable resource is used by a single economic sector, examples may be cited in which multiple

economic sectors are potential users of the resource. The extension of our basic model for the case of multiple economic sectors is carried in this section.

For this analysis we assume that each economic sector, $i=1,2,\ldots,I$, determines its costs of meeting the demand for end-use goods (2.6) in period t, $\Phi_i^t(r)$, by solving in each period $t=1,2,\ldots,T$ a sectoral problem. We further assume that $\Phi_i^t(0) < +\infty$ for $i=1,2,\ldots,I$ and $t=1,2,\ldots,T$, or, in other words, that the demand for end-use goods can be met by all economic sectors at a finite cost in the absence of the depletable resource. Let the cost-savings for sector i in period t be represented by

$$\beta_i^t(r) \equiv \Phi_i^t(0) - \Phi_i^t(r). \tag{3.1}$$

The function β_i^t would give, according to definition (2.8), the supplier's revenue if economic sector i were the unique user of the depletable resource.

We further assume that the economic sector i will be indifferent when exactly $\Phi_i^t(0) - \Phi_i^t(r_i)$ is paid in period t for r_i units of the resource. The supplier will then allocate a supply of r units among sectors in each period such as to maximize his revenue. This permits us to define the supplier's revenue function as the solution to the *multi-sectoral problem*

$$\begin{aligned} \beta^t(r) \equiv \max \ & \sum_{i=1}^{I} \beta_i^t(r_i) \\ \text{s.t.} \ & \sum_{i=1}^{I} r_i \le r \\ & r_i \ge 0 \quad i=1,2,\ldots,I \end{aligned} \tag{3.2}$$

and establish as an *equilibrium condition*:

"For any non-negative resource levels $r_i^1, r_i^2, \ldots, r_i^T$, $i=1,2,\ldots,I$ satisfying $\sum_{t=1}^{T} \sum_{i=1}^{I} r_i^t \le R$ we say that the sectoral problems and the supplier's problem are in equilibrium if these resource levels permit the supplier to maximize his profit; namely

$$V_T(R) = \sum_{t=1}^{T} \alpha^{t-1} \{ \sum_{i=1}^{I} [\Phi_i^t(0) - \Phi_i^t(r_i^t)] - g^t(\sum_{j=1}^{t-1} \sum_{i=1}^{I} r_i^j, \sum_i r_i^t) +$$

$$\alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} \sum_{i=1}^{I} r_i^t) \text{ ''}.$$

The properties of β^t, as given by the multi-sectoral problem (3.2), are summarized by the following lemmas, whose proofs are trivial and therefore omitted.

<u>Lemma 3.1</u>: If, for $i=1,2,\ldots,I$, β_i^t is concave in r_i, then β^t is

i) concave;

ii) non-decreasing in r.

<u>Lemma 3.2</u>: If β_i^t is piecewise linear in r_i with a finite number of segments, $i=1,2,\ldots,I$, then β^t is piecewise linear in r with a finite number of segments.

Other properties of β^t that can be readily verified are $\beta^t(0) = 0$ and continuity at $r = 0$ if the functions β_i^t are continuous at this point.

From Lemma 3.2, if, for all economic sectors, the sectoral problems are linear programs or grid linearizations of convex programs (see section 2.IV.1.), the revenue function β^t given by the multi-sectoral problem (3.2) will be piecewise linear.

The iterative approach of section 2.III. is easily extended to encompass the case of multiple economic sectors. We shall work with upper-bound approximations for β^t resulting from upper-bound approximations for the individual sectors' cost-savings functions

$$\tilde{\beta}_i^{t,n}(r) = \underset{k=1,2,\ldots,L^{t,n}}{\text{minimum}} \{b_i^{t,k} + \pi_i^{t,k} r\}. \tag{3.4}$$

This is the general form of the approximations (2.11) and (2.17). For the LP-sectoral problems (2.10), $L_i^{t,n} \le K_i^t$ and for the convex sectoral problems (2.16), $L_i^{t,n} < \infty$. An upper-bound approximation for the supplier's revenue function results by taking

$$\begin{aligned} \tilde{\beta}^{t,n}(r) = \max \sum_{i=1}^{I} \tilde{\beta}_i^{t,n}(r_i) \\ \text{s.t.} \sum_{i=1}^{I} r_i \le r \\ r_i \ge 0 \qquad i=1,2,\ldots,I\ . \end{aligned} \tag{3.5}$$

It is trivial to verify that $\tilde{\beta}^{t,n}(r) \ge \beta^t(r)$ for all $r \ge 0$. Furthermore, $\tilde{\beta}^{t,n}$ will be concave, non-decreasing and piecewise linear with a finite number of segments as a consequence of Lemmas 3.1 and 3.2.

At iteration n, the supplier's problem (2.1) is solved using the approximations $\tilde{\beta}^{t,n}$, resulting in a vector of non-negative resource levels

$r^{t,n}$ satisfying $\sum_{t=1}^{T} r^{t,n} \leq R$. These resource levels are sent to the multi-sectoral problems (3.5) to be disaggregated into the non-negative resource levels $r_i^{t,n}$ satisfying $\sum_{i=1}^{I} r_i^{t,n} \leq r^{t,n}$ for t=1,2,...,T. The sectoral problems are then optimized at these resource levels $r_i^{t,n}$, yielding the costs $\Phi_i^t(r_i^{t,n})$ and shadow prices $\pi_i^{t,n+1}$. If

$$\tilde{\beta}^{t,n}(r^{t,n}) = \sum_{i=1}^{I} \{\Phi_i^t(0) - \Phi_i(r_i^{t,n})\}$$

for all t=1,2,...,T, then $(r_i^{1,n}, r_i^{2,n}, \ldots, r_i^{T,n})$, i=1,2,...,I, is an equilibrium solution because it satisfies (3.3). Otherwise for at least one economic sector i and one time period t it must be that

$$\tilde{\beta}_i^{t,n}(r_i^{t,n}) > \Phi_i^t(0) - \Phi_i^t(r^{t,n}).$$

By adding the new linear segment

$$\Phi_i^t(0) - \Phi_i^t(r_i^{t,n}) + \pi_i^{t,n+1}(r_i - r_i^{t,n})$$

to the approximations (3.4), a tighter approximation, $\tilde{\beta}_i^{t,n+1}$, is obtained for sector i in period t. Consequently a tighter approximation results for the supplier's revenue function, $\tilde{\beta}^{t,n+1}$, from the inclusion of this new linear segment in the multi-sectoral problems (3.5). The supplier's problem (2.1) is then reoptimized using $\tilde{\beta}^{t,n+1}$ and new resource levels $r^{t,n+1}$ are sent back to the multi-sectoral problems. This extended scheme is depicted in Figure 3.1.

In the case the sectoral problems are linear programs, convergence to an equilibrium solution after a finite number of iterations would follow

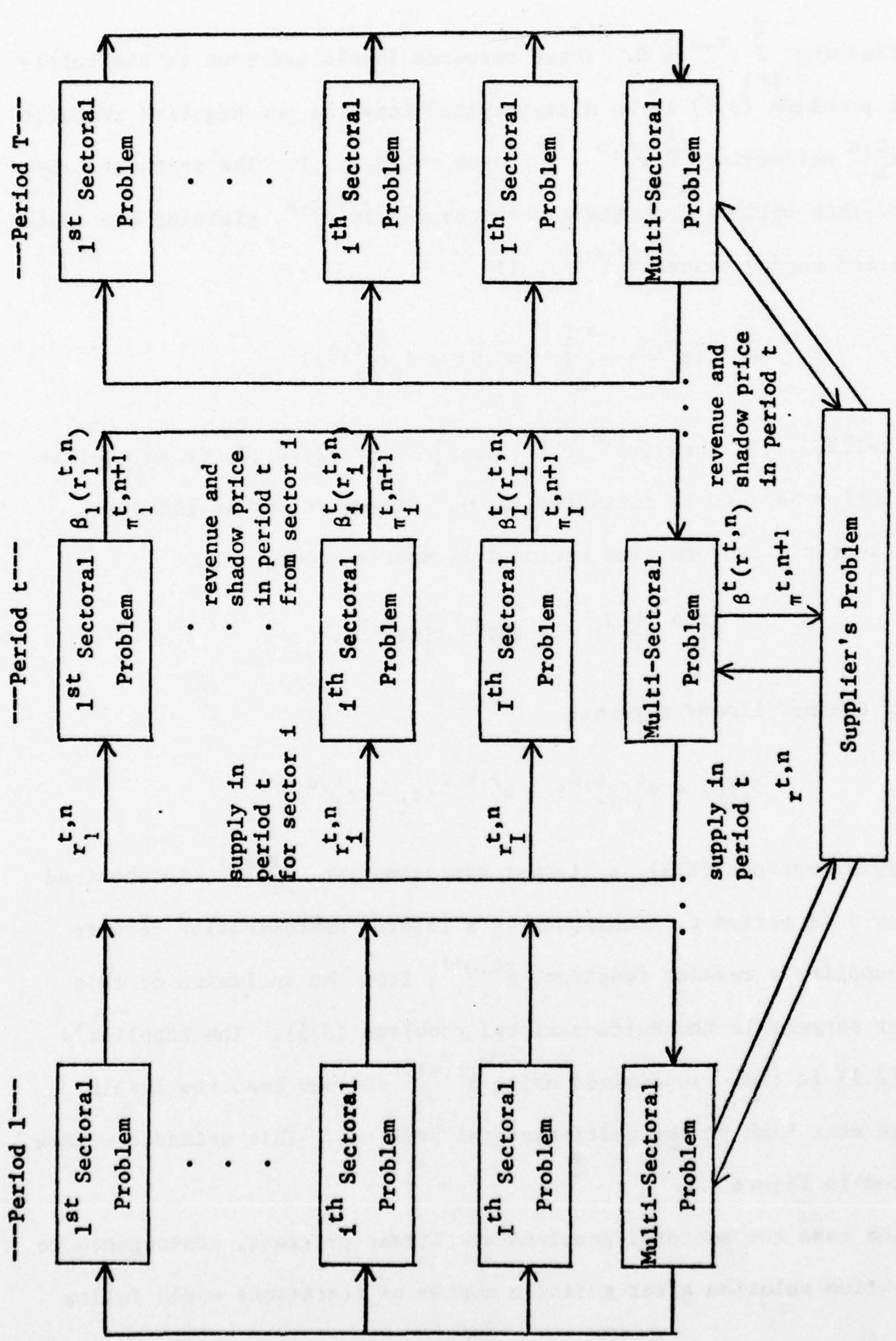

---Period 1---
---Period t---
---Period T---
1^{st} Sectoral Problem
i^{th} Sectoral Problem
I^{th} Sectoral Problem
Multi-Sectoral Problem
$r_1^{t,n}$
supply in period t for sector i
$r_i^{t,n}$
$r_I^{t,n}$
$\beta_1(r_1^{t,n})$
$\pi_1^{t,n+1}$
revenue and shadow price in period t from sector i
$\beta_i^t(r_i^{t,n})$
$\pi_i^{t,n+1}$
$\beta_I^t(r_I^{t,n})$
$\pi_I^{t,n+1}$
supply in period t
$r^{t,n}$
$\beta^t(r^{t,n})$
$\pi^{t,n+1}$
revenue and shadow price in period t
Supplier's Problem

Figure 3.1

from the finiteness of the number of economic sectors, the number of linear segments in β^t and the number of time periods. This argument is summarized in the following theorem whose proof is similar to that of Theorem 2.1 and therefore omitted.

Theorem 3.1: (Finite Convergence) The iterative scheme described above will converge to an equilibrium solution after a finite number of iterations between the multiple LP-sectoral problems (2.10) and the supplier's problem (2.1).

Under the more general convex specification (2.16) for the sectoral problem, if each sectoral problem, i=1,2,...,I, satisfies a Slater condition (see section 2.III.2.), convergence follows. The proof of this statement is also similar to that of Theorem 2.2 and is therefore omitted.

Theorem 3.2: (Infinite Convergence) The iterative scheme described above converges to an equilibrium between the multiple convex sectoral problems (2.16) and the supplier's problem (2.1) as the number of iterations tends to infinite.

We may observe that equilibrium as defined in (3.3) is for the linear programming formulation, in an analogous fashion to problem (2.15), a solution to

$$\max \sum_{t=1}^{T} \alpha^{t-1} \{ \sum_{i=1}^{I} \phi_i^t(0) - c_i^t x_i^t - f_i^t s_i^t - g(\sum_{j=1}^{t-1} r^j, r^t)\} +$$
$$\alpha^T \beta^{T+1} (R - \sum_{t=1}^{T} r^t)$$

$$
\begin{aligned}
\text{s.t.}\quad & \rho_i^t x_i^t \le r_i^t && t=1,2,\ldots,T,\ i=1,\ldots,I \\
& A_{1,i}^t x_i^t - s_i^t \le 0 && t=1,2,\ldots,T,\ i=1,\ldots,I \\
& A_{2,i}^t x_i^t \ge d_i^t && t=1,2,\ldots,T,\ i=1,\ldots,I \\
& \sum_{i=1}^{I} r_i^t \le r^t && t=1,2,\ldots,T \\
& \sum_{t=1}^{T} r^t \le R && \\
& 0 \le r_i^t,\ 0 \le s_i^t \le \bar{s}_i^t, && t=1,2,\ldots,T,\ i=1,2,\ldots,I \\
& 0 \le r^t && t=1,2,\ldots,T
\end{aligned}
\tag{3.6}
$$

The application of Benders' [2] decomposition to problem (3.6) will generate the above scheme. For fixed non-negative resource levels r_i^t, $t=1,2,\ldots,T$, $i=1,2,\ldots,I$ and r^t, $t=1,2,\ldots,T$, satisfying $\Sigma_{i=1}^{I} r_i^t \le r^t$ and $\Sigma_{t=1}^{T} r^t \le R$, problem (3.6) decomposes into IXT subproblems, the LP-sectoral problems (2.10). The master problem is

$$
\max_{r^t} \sum_{t=1} \alpha^{t-1} \left(\begin{aligned} & \sum_{i=1}^{I} \max_{r_i^t} \{\Phi_i^t(0) - \Phi_i^t(r_i^t)\} \\ & \text{s.t.} \sum_{i=1}^{I} r_i^t \le r^t \\ & 0 \le r_i^t,\ i=1,2,\ldots,I \end{aligned} \right) - g\left(\sum_{j=1}^{t-1} r^j, r^t\right) + \alpha^T \beta^{T+1} \left(R - \sum_{t=1}^{T} r^t\right)
$$

$$
\begin{aligned}
\text{s.t.}\quad & \sum_{t=1}^{T} r^t \le R \\
& r^t \ge 0 \qquad t=1,2,\ldots,T
\end{aligned}
$$

where the multi-sectoral problem (3.2) appears inside brackets. The ad-

vantage of including this intermediate step between the supplier's problem and the sectoral problem is the reduction in the number of variables in the supplier's problem. Instead of determining directly r_i^t, $i=1,2,\ldots,I$, $t=1,2,\ldots,T$ in a solution of the supplier's problem, only r^t, $t=1,2,\ldots,T$ is determined.

As the multi-sectoral problems are at each iteration the maximization of a sum of concave piecewise linear approximations (3.4) the disaggregation of the total depletable resource supply into the supplies for each economic sector requires only ordering the available π_i^t in descending order and allocating r^t into those blocks (see section 2.VII). The revenue function (3.2) results from integrating the horizontal summation of the individual economic sectors' marginal revenues as depicted below.

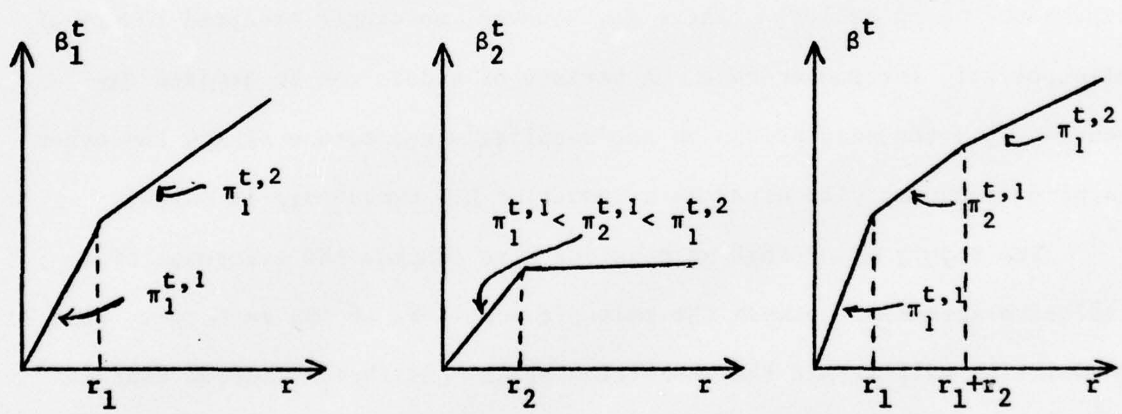


Figure 3.2

This section sets a framework in which sectoral or subsectoral submodels of the size of the current macro sectoral models (see BESOM [3], ETA [61]) can be integrated. This integration is accomplished essentially by shadow pricing mechanisms.

3.III. Multiple Depletable Resource Suppliers

The extension of the equilibrium model in Chapter 2 to cover the case of multiple suppliers is carried out in this section. This case is more complex because of the possible modes of behavior of the individual suppliers. The multiple suppliers of the homogeneous depletable resource may either collude or act independently. In independent action the key feature in the characterization of oligopoly markets is the recognized interdependence among sellers. There is, however, no single accepted theory of oligopolistic interdependence. A variety of models can be devised depending upon the assumptions on one supplier's conjecture of how the other supplier's output will alter as a result of his own change in output.

The supply model that we consider here assumes the existence of a collusive agreement between the multiple suppliers of the resource. This formulation will permit the definition of an equilibrium problem that is essentially similar to the sectoral-supplier equilibrium of Chapter 2. Alternative depletable resource oligopolistic supply models will be discussed in section 3.V.

For an homogeneous resource, we can define the revenue function of each individual supplier, $j=1,2,\ldots,J$, as

$$\beta_j^t(r_j, \sum_{\substack{i=1 \\ i\neq j}}^{J} r_i) = \beta^t(\sum_{i=1}^{J} r_i) \frac{r_j}{\sum_{i=1}^{J} r_i} \tag{3.7}$$

or in other words, each supplier receives its share of the joint revenue as

$$\sum_{j=1}^{J} \beta_j^t(r_j, \sum_{\substack{i=j \\ i\neq j}}^{J} r_i) = \beta^t(\sum_{i=1}^{J} r_j) \quad .$$

We can then show

<u>Lemma 3.3</u>: If β^t is concave, non-decreasing, continuous and satisfies $\beta^t(0) = 0$, then β_j^t, $j=1,2,\ldots,J$, is

(i) concave in r_j for fixed r_i, $i=1,2,\ldots,J$, $i\neq j$

(ii) non-decreasing in r_j for fixed r_i, $i=1,2,\ldots,J$, $i\neq j$

(iii) continuous in both arguments

(iv) $\beta_j^t(0, \sum_{\substack{i=1 \\ i\neq j}}^{J} r_i) = 0$ for all $r_i \geq 0$.

Proof: For notational simplicity we let

$$r_{-j} = \sum_{\substack{i=1 \\ i\neq j}}^{J} r_i \quad .$$

From the concavity of β^t we have

$$\beta^t(r_j+\hat{r}_{-j}) \leq \beta^t(\hat{r}_j+\hat{r}_{-j}) + \pi^t(r_j-\hat{r}_j)$$

where $\pi^t \geq 0$ is a subgradient of β^t at $r=\hat{r}_j+\hat{r}_{-j}$. It can be shown that

$$\beta^t(r_j+\hat{r}_{-j}) \frac{\hat{r}_j}{r_j+\hat{r}_{-j}} \leq \beta^t(\hat{r}_j+\hat{r}_{-j}) \frac{\hat{r}_j}{\hat{r}_j+\hat{r}_{-j}} +$$

$$+ \left[\pi^t \frac{\hat{r}_j}{\hat{r}_j+\hat{r}_{-j}} + \beta^t(\hat{r}_j+\hat{r}_{-j}) \frac{\hat{r}_{-j}}{(\hat{r}_j+\hat{r}_{-j})^2}\right] (\hat{r}_j-\hat{r}_j) +$$

$$+ (r_j-\hat{r}_j)^2 \frac{\hat{r}_{-j}}{(r_j+\hat{r}_{-j})\ (\hat{r}_j+\hat{r}_{-j})} \left[\pi^t - \frac{\beta^t(\hat{r}_j+\hat{r}_{-j})}{\hat{r}_j+\hat{r}_{-j}}\right]$$

Since β^t is concave and $\beta^t(0) = 0$, the last term is non-positive. By definition (3.7), we have

$$\beta_j^t(r_j,\hat{r}_j) \leq \beta_j^t(\hat{r}_j,\hat{r}_{-j}) + \left[\frac{\pi^t\hat{r}_j}{\hat{r}_j+\hat{r}_{-j}} + \beta^t(\hat{r}_j+\hat{r}_{-j}) \frac{\hat{r}_{-j}}{(\hat{r}_j+\hat{r}_{-j})^2}\right] (r_j-\hat{r}_j)$$

where the term inside brackets, π_j^t, is a subgradient of β_j^t at $r_j=\hat{r}_j$. This completes the proof of part (i). Part (ii) follows because $\pi_j^t \geq 0$. Parts (iii) and (iv) are readily verified from the definitions (3.7). ||

When the multiple suppliers of the depletable resource collude, they determine their individual supply schedules $(r_j^1,r_j^2,\ldots,r_j^T)$, $j=1,2,\ldots,J$, over a planning horizon T, by jointly maximizing the present value of profits. They solve the *multiple suppliers' problem*

$$V_T(R_1,R_2,\ldots,R_j) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta^t(r^t) - \sum_{j=1}^{J} g_j^t(\sum_{i=1}^{t-1} r_j^i, r_j^t)\} +$$

$$+ \alpha^T \beta^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t, R_2 - \sum_{t=1}^{T} r_2^t, \ldots, R_J - \sum_{t=1}^{T} r_J^t)$$

s.t.

$$\sum_{t=1}^{T} r_j^t \leq R_j \qquad j=1,2,\ldots,J$$

$$\sum_{j=1}^{J} r_j^t = r^t \qquad t=1,2,\ldots,T \tag{3.8}$$

$$r_j^t \geq 0 \qquad j=1,2,\ldots,J \quad t-1,2,\ldots,T$$

where g_j^t is the individual supplier's extraction cost functions, β^t is the joint revenue function and β^{T+1} is a salvage valuation the suppliers agreed upon. The assumptions on those functions are unchanged from section 2.II.

The solution to the multiple suppliers' problem (3.8) can also be computed by backward dynamic programming recursions similar to (2.5). The state space is now the J-dimensional vector $(S_1,S_2,\ldots,S_J)$ where S_j denotes the reserves of the depletable resource held by the j-th supplier, j=1,2,...,J. The multiple suppliers' problem (3.8) is solved by determining

$$V_T(R_1,R_2,\ldots,R_J) = V_T^1(R_1,R_2,\ldots,R_J)$$

where

$$V_T^t(S_1,S_2,\ldots,S_J) = \underset{\substack{0\le r_j\le S_j \\ j=1,2,\ldots,J}}{\text{maximum}} \{\beta^t(\sum_{j=1}^{J} r_j) - \sum_{j=1}^{J} g_j^t(R_j-S_j,r_j) +$$

$$+ \alpha\, V_T^{t+1}(S_1-r_1,S_2-r_2,\ldots,S_J-r_J)\}$$

$$t=T,T-1,\ldots,1$$

with

$$V_T^{T+1}(S_1,S_2,\ldots,S_J) = \beta^{T+1}(S_1,S_2,\ldots,S_J) .$$

Defining the multiple suppliers' *joint revenue function* as

$$\beta^t(r) \equiv \Phi^t(0) - \Phi^t(r) \tag{3.9}$$

we establish as an *equilibrium* condition

"For any non-negative resource levels $r_1^t,r_2^t,\ldots,r_J^t$, $t=1,2,\ldots,T$, satisfying $\sum_{t=1}^{T} r_j^t \le R_j$, $j=1,2,\ldots,J$, we say that the sectoral problems and the multiple suppliers' problem are in *equilibrium* if these resource levels permit the suppliers to jointly maximize their profits, namely (3.10)

$$V_T(R_1,R_2,\ldots,R_J) = \sum_{t=1}^{T} \alpha^{t-1}\{\phi^t(0) - \phi^t(\sum_{j=1}^{J} r_j^t) - \sum_{j=1}^{J} g_j^t(\sum_{i=1}^{t-1} r_j^i, r_j^t)\}$$

$$+ \alpha^T \beta^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t,\ R_2 - \sum_{t=1}^{T} r_2^t,\ldots,R_J - \sum_{t=1}^{T} r_J^t)."$$

Equilibrium as defined by (3.10) can be reached through the same iterative approach of section 2.III. At each iteration, the multiple suppliers' problem can be solved using the piecewise-linear upper-bound approximations (2.13) or (2.17) to the joint revenue functions (3.9). The joint supply schedule

$$r^{t,n} = \sum_{j=1}^{J} r_j^{t,n} \qquad t=1,2,\ldots,T$$

is then sent to the sectoral problems (2.10) or (2.16) to be priced out. The process is then repeated.

The convergence properties of the above scheme are unchanged from Theorems 2.1 and 2.2. For this reason we state without proof

Theorem 3.3: (Finite Convergence) The iterative scheme converges to an equilibrium solution after a finite number of iterations between the LP-sectoral problems (2.10) and the multiple suppliers' problem (3.8).

Theorem 3.4: (Infinite Convergence) The iterative scheme converges to an equilibrium between the convex sectoral problems (2.16) and the multiple suppliers' problem (3.8) as the number of iterations tends to infinity.

We consider now the case in which each individual supplier's extraction cost function is specified by a cumulative extraction cost function (2.20), namely where

$$g_j^t(E_j, r_j) = e_j(E_j + r_j) - e_j(E_j) \quad . \quad j=1,2,\ldots,J$$

A cumulative extraction cost function for pooled resource reserves can be determined by

$$e(E) = \min \sum_{j=1}^{J} e_j(E_j)$$

$$\text{s.t.} \quad \sum_{j=1}^{J} E_j \geq E \tag{3.11}$$

$$0 \leq E_j \leq R_j \qquad j=1,2,\ldots,J$$

It is easy to show that if each individual cumulative extraction cost function $e_j, j=1,\ldots, J$ is convex and increasing, the resulting joint cumulative extraction cost function is also convex and increasing. Also if each $e_j, j=1,2,\ldots,J$, is piecewise linear with a finite number of segments the resulting function e will also be piecewise linear with a finite number of segments.

By extending each individual cumulative extraction cost function to satisfy

$$e_j = +\infty \quad \text{for} \quad E_j > R_j \qquad j=1,2,\ldots,J$$

it is trivial to verify that e, given by (3.11) will result from the

integration of the horizontal summation of the individual marginal cumulative extraction costs as depicted below.

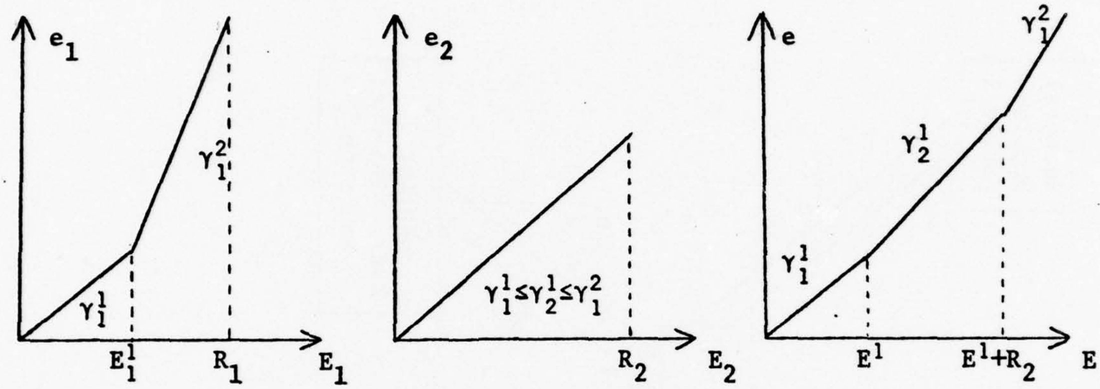


Figure 3.3

This observation suggests another iterative level in the solution of the multiple suppliers equilibrium (3.10) problem. The multiple suppliers' problem (3.8) could be solved at each iteration using also lower-bound approximations to the individual cumulative extraction cost functions $\underset{\sim}{e}_j$, that result either from the omission of some linear segments in the piecewise linear specification (2.20b) or tangential approximations to more general convex specifications. These lower-bound approximations will result in lower-bound approximations, $\underset{\sim}{e}$, to the aggregate or joint cumulative extraction cost function given by (3.11). Iteratively, they can be sequentially improved by the addition of new linear segments, simultaneously with the generation of new linear segments by the sectoral problems to the upper-bound approximations to the joint revenue function. This modified scheme is depicted in Figure 3.4.

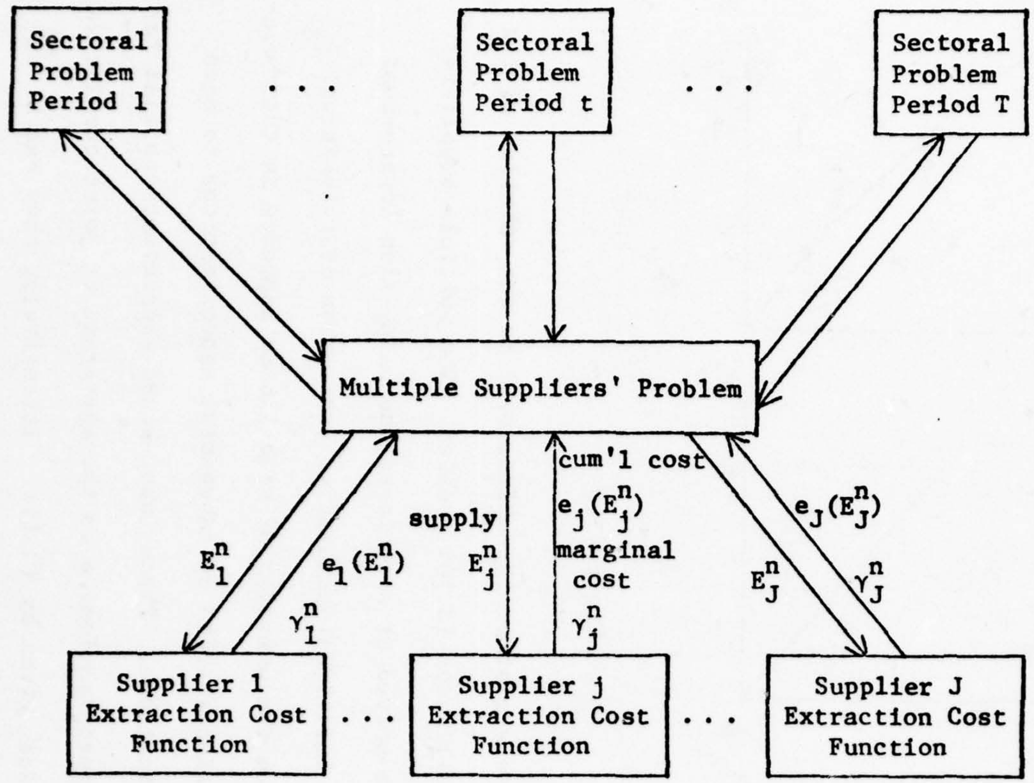

Sectoral Problem Period 1
Sectoral Problem Period t
Sectoral Problem Period T
Multiple Suppliers' Problem
E_1^n
$e_1(E_1^n)$
γ_1^n
supply
E_j^n
cum'l cost
$e_j(E_j^n)$
marginal cost
γ_j^n
$e_J(E_J^n)$
E_J^n
γ_J^n
Supplier 1 Extraction Cost Function
Supplier j Extraction Cost Function
Supplier J Extraction Cost Function

Figure 3.4

The advantage introduced by this parallel iterative approach is more clearly seen the larger the number of suppliers and the larger the number of linear segments in each individual cumulative extraction cost functions.

3.IV. Purely Competitive Depletable Resource Markets

In this section we assume that the depletable resource stock R is owned by a large number of individual suppliers (or firms) that individually control an amount R_i of the total reserves $(R = \sum_i R_i)$. Furthermore we assume that the market for the depletable resource is competitive: each individual supplier is a price-taker and maximizes the net present value of profits resulting from selling decisions. This assumption allows the individual supplier to determine an optimal supply schedule, $r_i^1, r_i^2, \ldots, r_i^T$, over the planning horizon T, by solving the *pure competitor supplier's problem*

$$V_{i,T}(R_i) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\hat{p}^t r_i^t - g_i^t(\sum_{j=1}^{t-1} r_i^j, r_i^t)\} + \alpha^T \hat{p}^{T+1} [R_i - \sum_{t=1}^{T} r_i^t]$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r_i^t \leq R_i \tag{3.12}$$

$$r_i^t \geq 0 \qquad t=1,2,\ldots,T$$

where $\hat{p}^t$ is the price per-unit of depletable resource sold and g_i^t is the extraction cost function. Stocks left at the end of the planning horizon

are valued at a price $\hat{p}^{T+1}$ per unit[1]. We assume that the individual extraction cost functions are convex and increasing. The objective in (3.15) is then concave and for differentiable extraction cost functions g_i^t, the Kuhn-Tucker conditions for optimality are necessary and sufficient. These require the existence of a pair $(\hat{r}_i, \hat{\lambda}_i)$ satisfying:

$$\hat{p}^t - \frac{dg_i^t}{dr_i^t} - \sum_{j=t+1}^{T} \alpha^{j-t} \frac{dg_i^j}{dr_i^t} - \hat{\lambda}_i \alpha^{1-t} - \alpha^{T-t+1} \hat{p}^{T+1} \leq 0 \qquad t=1,2,\ldots,T \qquad (3.13a)$$

$$(\hat{p}^t - \frac{dg_i^t}{dr_i^t} - \sum_{j=t+1}^{T} \alpha^{j-t} \frac{dg_i^j}{dr_i^t} - \hat{\lambda}_i \alpha^{1-t} - \alpha^{T-t+1} \hat{p}^{T+1})\, r_i^t = 0 \qquad t=1,2,\ldots,T \qquad (3.13b)$$

$$\sum_{t=1}^{T} \hat{r}_i^t \leq R_i \ , \ \hat{r}_i^t \geq 0 \qquad t=1,2,\ldots,T \qquad (3.13c)$$

$$\hat{\lambda}_i \left(\sum_{t=1}^{T} \hat{r}_i^t - R_i \right) = 0 \qquad (3.13d)$$

$$\hat{\lambda}_i \geq 0 \qquad (3.13e)$$

The interpretation of (3.13c), (3.13d) and (3.13e) is trivial. For $\hat{r}_i^t > 0$, (3.13b) gives

[1] An explanation for the linearity of the individual supplier's salvage function is the existence of a parallel competitive market for stocks as suggested by Stiglitz [85].

$$\hat{p}^t - \frac{dg_i^t}{dr_i^t} = \sum_{j=t+1}^{T} \alpha^{t-j} \frac{dg_i^j}{dr_i^t} + \hat{\lambda}_i \alpha^{1-t} - \alpha^{T-t+1} \hat{p}^{T+1} \tag{3.14}$$

The term in the right-hand side of (3.14) has been interpreted for (2.3). The left-hand term is the difference between price and the marginal immediate cost of the i-th supplier. Equation (3.14) then requires for the i-th competitor that price equals total marginal costs (immediate plus total user's costs).

Similarly to the development in section 2.II the pure competitor supplier's problem (3.12) can be reformulated as a backward dynamic programming problem. The backward recursions are similar to (2.5). When the individual competitive supplier's extraction cost function is piecewise linear as given by (2.20), the pure competitor supplier's problem can be transformed into a linear program analogous to (2.23). The LP-pure competitor supplier's problem will involve less constraints than (2.23) because the revenue function and salvage function in (3.12) are linear functions of the supply in each period t and of the iddle stock at the end of the planning horizon respectively.

However, we shall abstract from further considerations on the individual supplier's problem and concentrate on the aggregate supply of the competitive market. Considering the depletable resource homogeneous, the economic sector responds only to total quantities (aggregate supply) in each period, irrespectively of its individual composition. For this purpose we let p^t denote the economic sector demand function for the depletable resource in period t. We further assume p^t is non-increasing, and

that the integral

$$P^t(r) = \int_0^r p^t\ (\zeta)\ d\zeta \qquad (3.15)$$

exists.[1] Then by Rudin [ ]

$$p^t(r) = \frac{d}{dr}\left[\int_0^r p^t\ (\zeta)\ d\zeta\right].$$

The area under the demand curve in (3.15) can be interpreted as a measure of the sectoral surplus. Similarly to the concept of consumer's surplus (see Samuelson [72]) when p^t refers to consumer demand, (3.15) is a measure of the benefit to the economic sector of having r units of the depletable resource available in period t.

We can then show

Lemma 3.4: If p^t is non-increasing, and (3.15) exists then $P^t(r)$ is concave.

Proof: Since p^t is non-increasing,

$$[p^t(r) - p^t(r_o)]\ (r - r_o) \leq 0$$

Therefore, for $\zeta \geq r_o$ we have

[1]A sufficient condition for (3.15) to exist is boundedness and continuity (see Rudin [71].)

$$\int_{r_o}^{r} p^t(\zeta)\, d\zeta \le \int_{r_o}^{r} p^t(r_o)\, d\zeta$$

which yields

$$P^t(r) \le P^t(r_o) + p^t(r_o)\,(r-r_o)\ .$$

Conversely for $\zeta \le r_o$ we have

$$\int_{r}^{r_o} p^t(\zeta)\, d\zeta \ge \int_{r}^{r_o} p^t(r_o)\, d\zeta$$

also yielding

$$P^t(r) \le P^t(r_o) + p^t(r_o)\,(r-r_o)$$

Consequently P^t is concave. ||

Similarly to static competitive equilibrium resulting from the maximization of consumers' plus producers' surplus we shall argue that an intertemporal competitive equilibrium will result from the maximization of the constrained discounted sectoral plus suppliers' surplus over the planning horizon T. The *competitive market problem* is then

$$\max \sum_{t=1}^{T} \alpha^{t-1} \left\{ \int_0^{r^t} \left[p^t(\zeta)\, d\zeta \right] - g^t\left(\sum_{j=1}^{t-1} r^j, r^t\right) \right\} + \alpha^T \beta^{T+1}\left(R - \sum_{t=1}^{T} r^t\right)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r^t \leq R \tag{3.16}$$

$$r^t \geq 0 \qquad t=1,2,\ldots,T$$

where $-g^t$ is the suppliers' (producers') surplus (the integral of the aggregate supply function) and β^{T+1} is a salvage function the market agrees upon[1]. We assume g^t is convex and non-decreasing and β^{T+1} is concave, non-decreasing and satisfying $\beta^{T+1}(0) = 0$.

The Kuhn-Tucker necessary and sufficient conditions for problem (3.16) require the existence of a pair $(\hat{r},\hat{\lambda})$ satisfying

$$p^t - \frac{dg^t}{dr^t} - \sum_{j=t+1}^{T} \alpha^{j-t} \frac{dg^j}{dr^t} - \hat{\lambda}\, \alpha^{1-t} - \alpha^{T-t+1} \frac{d\beta^{T+1}}{dr^t} \leq 0$$

$$t=1,2,\ldots,T \tag{3.17a}$$

$$\left(p^t - \frac{dg^t}{dr^t} - \sum_{j=t+1}^{T} \alpha^{j-t} \frac{dg^j}{dr^t} - \hat{\lambda}\, \alpha^{1-t} - \alpha^{T-t+1} \frac{d\beta^{T+1}}{dr^t}\right) \hat{r}^t = 0$$

$$t=1,2,\ldots,T \tag{3.17b}$$

$$\sum_{t=1}^{T} \hat{r}^t \leq R \ , \ \hat{r}^t \geq 0 \qquad t=1,2,\ldots,T \tag{3.17c}$$

$$\hat{\lambda}\left(\sum_{t=1}^{T} \hat{r}^t - R\right) = 0 \tag{3.17d}$$

$$\hat{\lambda} \geq 0 \tag{3.17e}$$

---

[1] If a competitive market for reserves exists in period T+1, then

$$\beta^{T+1}(S) = \int_0^S p^{T+1}(\zeta)\, d\zeta$$

Weinstein and Zeckhauser [91] argued in their infinite-horizon model with cumulative extraction cost functions e_i, the efficiency of the competitive market, as it yields the same allocation as the social optimum determined by the solution of the infinite-horizon version of the competitive market problem (3.16). In case the aggregate cumulative extraction cost function is given by

$$e(E) = \min \sum_i e_i(E_i)$$

$$\Sigma E_i \geq E$$

$$0 \leq E_i \leq R_i \qquad \text{all } i$$

the aggregate marginal extraction cost function would be the horizontal summation of the individual marginal extraction cost functions. Consequently if the individual competitive suppliers are faced with the sequence of prices

$$\hat{p}^t = p^t(\hat{r}^t) \qquad t=1,2,\ldots,T$$

$$\hat{p}^{T+1} = \left.\frac{d\beta^{T+1}}{dS}\right|_{S=R-\sum_{t=1}^{T} \hat{r}^t}$$

where $(\hat{r}^1,\hat{r}^2,\ldots,\hat{r}^T)$ is an optimal solution to (3.16), the identity between the optimality conditions (3.13) and (3.17) can be readily verified, ascertaining that the same argument is valid here.

We consider now the case, where, the economic sector, on the depletable resource demand side, also acts as a price-taker. Faced in period t with a price $\hat{p}^t$, the sector determines its resource usage by solving in each period

$$\min \Phi^t(r) + \hat{p}^t r \qquad (3.18)$$

$$\text{s.t.} \quad r \geq 0$$

A necessary and sufficient condition for optimality in problem (3.18) requires that

$$\hat{p}^t + \frac{d\Phi^t}{dr} \geq 0 \qquad = 0 \text{ if } r \geq 0$$

This would give the *sectoral demand function* as

$$p^t = -\frac{d\Phi^t}{dr} \qquad (3.19)$$

for $r > 0$. Fig. 3.5, below, depicts the sectoral demand function that results when Φ^t is computed from the LP-sectoral problems (2.10).

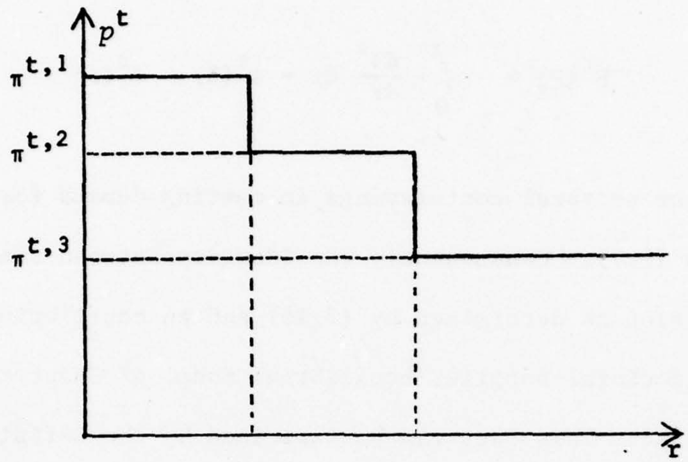


Figure 3.5

Fig. 3.6, in contrast, depicts a smooth demand function that could result from more general convex specifications for the sectoral problems such as (2.16).

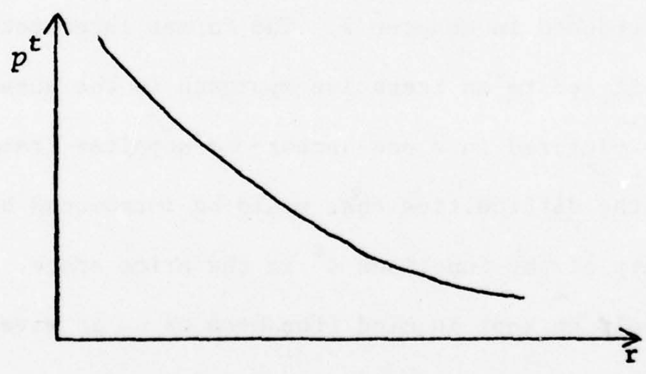


Figure 3.6

A competitive equilibrium can then be determined by solving the competitive market problem (3.16). Notice, however, that the sectoral surplus

$$P^t(r) = \int_0^r - \frac{d\Phi^t}{dr}\, dr = \Phi^t(0) - \Phi^t(r)$$

is given by the sectoral cost-savings in meeting demand for end-use goods as defined in (2.7). Consequently the identity between the efficient resource allocation as determined by (3.16) and an equilibrium solution (2.9) to the sectoral-supplier equilibrium model of Chapter 2 can be readily verified. This fact can be explained by the definitions (2.8) for the supplier's revenue functions in Chapter 2.

If the economic sector is a competitive price-taker, then an unique supplier capturing as revenue the full amount of the sectoral surplus will produce the competitive efficient resource allocation over time. Therefore we have two alternative interpretations for the basic equilibrium model introduced in Chapter 2. The former interpretation is preferred because it led to an iterative approach in the quantity space that is more clearly pictured in a one-sector-one-supplier framework. It also avoids much of the difficulties that would be introduced by the non-differentiability of the functions Φ^t in the price space. However this equivalence should be kept in mind from here on as it gives an alternative interpretation to the extensions carried out in subsequent chapters.

The unique distinction that needs to be made between the competitive equilibrium and the basic equilibrium (2.9) concerns the resource prices. For a unique supplier receiving the sectoral cost-savings as revenue his receipts per unit would be given by

$$\frac{\beta^t(r)}{r} \equiv \frac{\phi^t(0) - \phi^t(r)}{r}$$

as depicted in Figure 3.7.

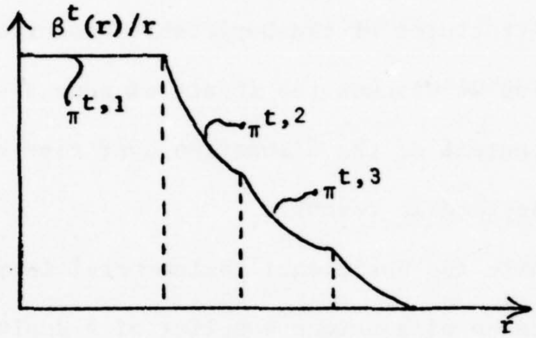


Figure 3.7

In contrast, competitive equilibrium prices would be determined by

$$p^t(r) = \frac{d\beta^t}{dr}$$

as depicted in Figure 3.5.

A few comments are in order. Some of the extensions of the basic equilibrium model carried out in section 2.V.1 may need a different treatment in the present context. One such example is the case of price regulation. As price regulation would limit the prices that can prevail in competitive equilibrium we would have here as the driving force of the modified equilibrium, the restraint

$$\hat{\pi}^{t,n+1} = \min \{\bar{p}^t, \pi^{t,n+1}\}$$

instead. Another point worth mentioning are the results of section 2.VI

for finite versus infinite time depletion. These could be reinterpreted here in terms of whether or not the sectoral demand function (3.19) intercepts the vertical axis.

3.V. Alternative Structures of the Depletable Resource Supply Market

In this section we discuss the impact of some alternative market structures in the context of the allocation over time of the fixed resource stock of a depletable resource.

Monopoly. While the basic equilibrium model introduced in Chapter 2 assumed the existence of a unique supplier of a depletable resource, the observations of section 3.IV confirm that this was the case of a perfectly discriminating monopoly (see Malinvaud [60]). Here, instead, we consider briefly the traditional monopoly framework, where the monopolist faces the sectoral demand function (3.19). The monopolist is assumed to determine a supply schedule $r^1, r^2, \ldots, r^T$ over a planning horizon T by solving the *monopolist problem*

$$\max \sum_{t=1}^{T} \alpha^{t-1} \{p^t(r^t)r^t - g^t(\sum_{j=1}^{t-1} r_j, r_t)\} + \alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} r^t)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r^t \leq R \tag{3.20}$$

$$r^t \geq 0 \qquad t=1,2,\ldots,T$$

with the cost functions ϕ^t being derived from the LP-sectoral problems (2.10). The shape of the monopolist total revenue is depicted below.

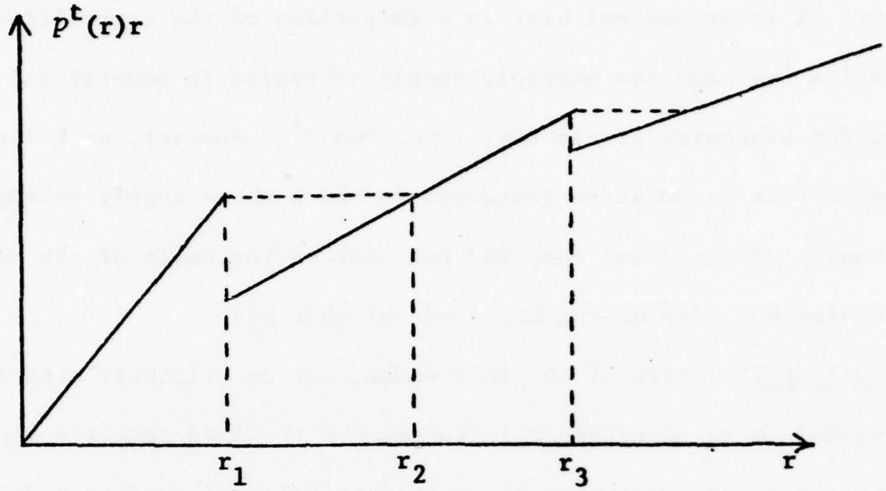


Figure 3.8

This situation illustrates an anomaly of demand functions derived from linear programs. Obviously the monopolist will not be willing to supply the resource in the range (r_1, r_2) as his total receipts would be lower than selling to the economic sector r_1 units and disposing of the remaining $r-r_1$ units. Hence for all purposes in solving (3.20) we can restrict ourselves to the continuous function in dashed lines in Figure 3.8. However the monopolist problem's objective will not be concave and furthermore non-differentiable optimization techniques would be required to solve (3.20) (see Zangwill [95]).

While Sweeney [87] and Weinstein and Zeckhauser [91] have shown that for constant elasticity demand functions, the competitive and monopolist supply schedules with no extraction costs would coincide and that for linear demand functions the monopolist will overconserve the resource, no a priori conclusions can be easily drawn in our case. Apparently the

question of intertemporal bias in a comparison of the competitive market of section 3.IV and the monopoly supply schedules is non-trivial to answer for piecewise linear cost functions Φ^t. However, we believe to have set the computation framework in which these supply schedules can be actually computed and compared not only on the basis of the direction of the bias but also on the magnitude of this bias.

Oligopoly. Most of the independent action oligopoly theories can be extended to an intertemporal framework with fixed resource stocks. However the solution of even the simple traditional duopoly models becomes quite complicated when at the same time we introduce simultaneously non-differentiable revenue functions, extraction cost functions depending upon cumulative extraction and a finite planning horizon. In static Cournot [9] duopoly theory each individual supplier using its individual revenue function maximizes profits, assuming that the other oligopolists will not alter their output as a result of his decision. In a duopoloy with depletable resources, duopolist i (i=1,2) solves the *Cournot duopolist problem*

$$V_T^i(R_i;r_j^1,r_j^2,\ldots,r_j^T) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^t,r_j^t) - g_i^t(\sum_{k=1}^{t-1} r_i^k,r_i^t)\} +$$

$$\alpha^T\beta_i^{T+1}(R_i - \sum_{t=1}^{T} r_i^t)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r_i^t \le R_i \qquad (3.21)$$

$$r_i^t \ge 0 \quad t=1,2,\ldots,T$$

assuming $(r_j^1, r_j^2, \ldots, r_j^T)$ fixed, $j \neq i$, and with β_i^t defined by (3.7). Let the set of optimal solutions to (3.21) be characterized by

$$V^i(r_j) = \left\{ r_i \;\middle|\; r_i^t \geq 0, t=1,2,\ldots,T, \sum_{t=1}^{T} r_i^t \leq R_i \quad \text{and} \right.$$

$$\sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^t, r_j^t) - g_i^t(\sum_{k=1}^{t-1} r_i^i, r_i^t)\} +$$

$$\left. + \alpha^T \beta^{T+1}(R_i - \sum_{t=1}^{T} r_i^t) = V_T^i(R_i; r_j) \right\} \tag{3.22}$$

where for notational simplicity $r_i = (r_i^1, r_i^2, \ldots, r_i^T)$, for $i=1,2$ and $j \neq i$. This set is a generalization of the concept of reaction function (see Intrilligator[47]) to non-strictly concave objectives and multiperiod planning. It gives the reaction set of duopolist i to fixed supply schedules of duopolist j in the planning horizon T ($i=1,2$ and $j \neq i$).

The pair $(\hat{r}_1, \hat{r}_2)$ is defined to be a *Cournot equilibrium* if

$$\left\{ \begin{array}{l} \hat{r}_1 \in V^1(\hat{r}_2) \\ \hat{r}_2 \in V^2(\hat{r}_1) \end{array} \right. \tag{3.23}$$

<u>Lemma 3.5</u>: The image of $V^i(r_j)$ is a

(i) convex

(ii) closed

subset of $X_i = \{r_i \geq 0 \mid \sum_{t=1}^{T} r_i^T \leq R_i\}$

Proof: Let $r_i^1, r_i^2 \; \varepsilon \; V^i(r_j)$. Then for any $0 \leq \omega \leq 1$

$$\omega r_i^{t,1} + (1-\omega)\, r_i^{t,2} \geq 0 \qquad\qquad t=1,2,\ldots,T$$

and

$$\sum_{t=1}^{T} \omega r_i^{t,1} + (1-\omega)\, r_i^{t,2} \leq R\, .$$

Consequently,

$$\begin{aligned}
V_T^i(R_i;r_j) \geq \sum_{t=1}^{T} \alpha^{t-1} \{ & \beta_i^t(\omega r_i^{t,1} + (1-\omega) r_i^{t,2}, r_j^t) - \\
& g_i^t(\sum_{k=1}^{t-1} [\omega r_i^{k,1} + (1-\omega) r_i^{k,2}],\, \omega r_i^{t,1} + (1-\omega)\, r_i^{t,2}) + \\
& \alpha^T \beta_i^{T+1} (R_i - \sum_{t=1}^{T} [\omega r_i^{t,1} + (1-\omega)\, r_i^{t,2}]) \geq \\
& \omega \left[\sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,1}) - g^t(\sum_{k=1}^{t-1} r_i^{k,1},\, r_i^{t,1})\} + \right. \\
& \left. \alpha^T \beta_i^{T+1} (R_i - \sum_{t=1}^{T} r_i^{t,1}) \right] + (1-\omega) \left[\sum_{t=1}^{T} \alpha^t \{\beta_i^t(r_i^{t,2}) - \right. \\
& \left. g_i^t(\sum_{k=1}^{t-1} r_i^{k,2}, r_i^{t,2}) + \alpha^T \beta_i^{T+1} (R_i - \sum_{t=1}^{T} r_i^{t,2}) \right] = V_T^i(R_i;r_j)
\end{aligned}$$

where the first inequality follows since $\omega r_i^1 + (1-\omega)r_i^2$ is feasible for (3.21) and the second inequality from the concavity of the objective. We conclude that

$$\omega r_i^1 + (1-\omega)\, r_i^2 \in V^i(r_j)$$

which completes the proof of (i).

Let $\{r_i^n\}$ be a sequence in $V^i(r_j)$ converging to r_i^*. We have for all terms in the sequence

$$\sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,n}) - g_i^t(\sum_{k=1}^{t-1} r_i^{k,n},\ r_i^{t,n})\} + \alpha^T \beta^{T+1}(R_i - \sum_{t=1}^{T} r_i^{t,n}) =$$

$$V_T^i(R_i;r_j) \quad .$$

From the continuity of the objective, we have after taking limits

$$\sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,*}) - g_i^t(\sum_{k=1}^{t-1} r_i^{k,*}, r_i^{t,*})\} + \alpha^T \beta^{T+1}(R_i - \sum_{t=1}^{T} r_i^{t,*}) =$$

$$V_T^i(R_i;r_j) \ .$$

Therefore,

$$r_i^* \in V^i(r_j)$$

which completes the proof of the lemma.||

<u>Lemma 3.6</u>: The point-to-set mapping $V^i(r_j)$ is upper semi-continuous.

Proof: Let $\{r_i^n\}$ and $\{r_j^n\}$ be sequences such that $r_i^n \in V^i(r_j^n)$ converging to r_i^* and r_j^* respectively. Clearly,

$$r_i^{t,*} \geq 0 , \qquad t=1,2,\ldots,T$$

and

$$\sum_{t=1}^{T} r_i^{t,*} \leq R_i$$

since X_i is a compact set.

Consequently,

$$V_T^i(R_i;r_j^*) \geq \sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,*},r_j^{t,*}) - g_i^t(\sum_{k=1}^{t-1} r_i^{k,*}, r_i^{t,*})\} +$$

$$\alpha^T\beta^{T+1}(R_i - \sum_{t=1}^{T} r_i^{t,*}) =$$

$$\lim \sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,n},r_j^{t,n}) - g_i^t(\sum_{k=1}^{t-1} r_i^{k,n},r_i^{t,n})\} +$$

$$\alpha^T\beta_i^{T+1}(R_i - \sum_{t=1}^{T} r_i^{t,n}) =$$

$$\lim V_T^i(R_i;r_j^n)$$

where the inequality follows because r_i^* is feasible for (3.21) at r_j^*, the first equality from the continuity of the objective and the second equality by definition.

All it remains to show is that

$$\lim V_T^i(R_i;r_j^n) \geq V_T^i(R_i;r_j^*) \tag{3.24}$$

or in other words, that V_T^i is a lower semi-continuous function of r_j. Suppose the contrary,

$$\lim V_T^i(R_i;r_j^n) < V_T^i(R_i;r_j^*)$$

then for n sufficiently large and any $r_i^o \in V^i(r_j^*)$ from the continuity of the objective we would have

$$\sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,n},r_j^{t,n}) - g_i^t(\sum_{k=1}^{t-1} r_i^{k,n},r_i^{t,n})\} + \alpha^T\beta_i^{T+1}(R_i - \sum_{t=1}^{T} r_i^{t,n}) <$$

$$\sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^{t,o},r_j^{t,n}) - g_i^t(\sum_{k=1}^{t-1} r_i^{k,o},r_i^{t,o})\} + \alpha^T\beta_i^{T+1}(R_i - \sum_{t=1}^{T} r_i^{t,o})$$

But this would contradict, for n sufficiently large, the assumption that

$$r_i^n \in V^i(r_j^n)$$

Hence we must have that (3.24) holds and consequently equality follows, giving

$$r_i^* \in V^i(r_j^*)$$

which proves the upper semi-continuity of the mapping V^i, as desired.||

Theorem 3.5: For a duopoly with depletable resources, there exists a Cournot equilibrium, as defined in (3.23).

Proof: Consider the point-to-set mapping

$$V(r_1,r_2) = [V^1(r_2),\ V^2(r_1)] \quad .$$

By Lemma 3.5, V maps (X_1,X_2) into a closed convex subset of (X_1,X_2). The point-to-set mapping is upper semi-continuous as a result of Lemma 3.6. Katukani's [48] fixed-point theorem (see Stoer & Witzgall [86]) guarantees the existence of a point $(\hat{r}_1,\hat{r}_2)$ such that

$$(\hat{r}_1,\hat{r}_2) \in V(\hat{r}_1,\hat{r}_2) \ ,$$

or equivalently, from the definition of V

$$\hat{r}_1 \in V^1(\hat{r}_2)$$

$$\hat{r}_2 \in V^2(\hat{r}_1)$$

which is a point of Cournot equilibrium by definition (3.23).||

In the spirit of this thesis, an equilibrium between the economic sector and the Cournot duopoly could be defined. An iterative scheme

between the economic sector and the Cournot duopoly would generate a sequence of tentative supply schedules $(r_1^1+r_2^1, r_1^2+r_2^2, \ldots, r_1^T+r_2^T)$. At each iteration the Cournot equilibrium could be solved by any method applicable for approximating fixed points (e.g. Scarf & Hansen [74], Scarf [73]).

We illustrate alternative models of oligopolistic reaction extended to the case of multiperiod planning with depletable resources. Proofs of existence of equilibria are similar to the one just given and will be omitted. Stackelberg's [83] leader-follower duopoly theory assumes that the leader (i=1) is sophisticated enough to take into account the reaction (3.22) of the follower, while the follower (i=2) solves problem (3.21). In a duopoly with depletable resource *Stackelberg's leader's problem* is to

$$V_T(R_1) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta_1^t(r_1^t, r_2^t) - g^t(\sum_{j=1}^{t-1} r_1^j, r_1^t) + \alpha^T \beta_1^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r_1^t \leq R_1$$

$$(r_2^1, r_2^2, \ldots, r_2^T) \in V^2(r_1^1, r_1^2, \ldots, r_1^T) \tag{3.25}$$

$$r_1^t \geq 0 \qquad t=1,2,\ldots,T$$

Other behavioral assumptions for the independent action duopolists stem from the market shares model, in which one duopolist (i=2) wishes to maintain a particular market share k, independent of the short-run effects on its profits, in other words

$$r_2^t = \frac{k r_1^t}{1-k} \qquad t=1,2,\ldots,T \qquad (3.26)$$

The other duopolist (i=1) would then maximize his profits utilizing the knowledge of (3.26), by substituting it for V^2, t=1,2,...,T, in (3.25).

Duopolistic behavior can also be transformed into a zero-sum game by Bishop's [4] warfare theory. It assumes that each duopolist maximizes the excess of its profits over the other duopolists profits. Consequently, the second duopolist acts as to minimize the first duopolist maximum excess profits.

Which of these duopoly solutions is more realistic is an unanswered question. There does not seem to be enough evidence that would in general support one model among all possibilities. For this reason, little progress has been made in oligopoly theory in recent years, and there seems to be a general trend to develop oligopolistic market analysis departing from the traditional theory's assumptions on the conjectural interdependence (see Cohen and Cyert [6]). These in general do not permit a mathematical programming treatment and shall not be exploited here.

3.VI. Multiple Depletable Resources

We consider in this section an economic sector whose primary supplies include multiple depletable resources. Denoting by r_m, the amount of depletable resource, m=1,2,...,M, available to the sector for the production of end-use goods, a model of the economic sector provides in each period the *cost function*

$$\Phi^t(r_1,r_2,\ldots,r_M) = \text{minimum cost of meeting demand for end-use goods in period } t\text{, when } r_m,\ m=1,2,\ldots,M, \text{ are the quantities of the M depletable resources available to the sector in period } t. \tag{3.27}$$

We assume as before that Φ^t is non-increasing in r_m, m=1,2,...,M and convex. We also require that $\Phi^t(0,0,\ldots,0) < +\infty$, or in other words, that the end-use goods demand can be met in each period (at finite cost) in the absence of the M depletable resources. This assumption permits us to define the *cost-savings function*

$$\Phi^t(0,0,\ldots,0) - \Phi^t(r_1,r_2,\ldots,r_M) = \text{cost-savings in meeting the demand for end-use goods in period } t \text{ when } r_m, m=1,2,\ldots,M \text{ are the quantities of depletable resource available to the sector in period } t. \tag{3.28}$$

If Φ^t is non-increasing and convex, the cost-savings function is non-decreasing and concave.

In an analogous fashion to Chapter 2, we illustrate here the computation of the cost-savings function (3.28) for two possible specifications of sectoral problems:

i) A LP-Sectoral Problem

$$\Phi^t(r_1, r_2, \ldots, r_m) = \min\ c^t x^t + f^t s^t$$

$$\text{s.t.} \quad \rho_m x^t \le r_m \qquad m=1,2,\ldots,M \quad (3.29a)$$

$$A_1^t x^t - s^t \le 0 \qquad (3.29b)$$

$$A_2^t x^t \ge d^t \qquad (3.29c)$$

$$0 \le x^t \text{ and } 0 \le s^t \le \mathit{s}^t \qquad (3.29d)$$

The properties of Φ^t, as given by problem (3.29) are summarized by the following lemma, whose proof is similar to that of Lemma 2.1 and therefore omitted.

Lemma 3.7: Φ^t given by (3.29) is

(i) non-increasing and convex in r_m, m=1,2,...,M

(ii) piecewise linear in r_m, m=1,2,...,M with a finite number of segments.

Linear programming duality theory would give the cost-savings function

(3.28) in an analogous fashion to (2.11) as

$$\Phi^t(0,0,\dots,0) - \Phi^t(r_1,r_2,\dots,r_M) = \underset{k=1,\dots,K^t}{\text{minimum}} \{(\Phi^t(0)-u^{t,k}\, d^t + w^{t,k} s^t) + \sum_{m=1}^{M} \pi_m^{t,k}\, r_m\}$$

ii) A Convex Sectoral Problem

$$\Phi^t(r_1,r_2,\dots,r_m) = \min\ c^t(x^t) + f^t(s^t)$$

$$\text{s.t. } F^t(d^t;\{x^t,s^t,z_1^t,z_2^t,\dots,z_M^t\}) \le 0 \qquad (3.30a)$$

$$z_m^t \le r_m \qquad (3.30b)$$

$$m=1,2,\dots,M$$

$$0 \le x^t\ ,\ 0 \le s^t \le s^t \qquad (3.30c)$$

Under the convexity conditions on c^t, f^t and F^t imposed in section 2.III, the properties of Φ^t generated by (3.30) are similar to those in Lemma 2.2, and are summarized without proof in the following lemma.

<u>Lemma 3.8</u>: Φ^t given by (3.30) is non-increasing and convex in r_m, $m=1,2,\dots,M$.

If problem (3.30) satisfies a Slater condition as in section 2.III, then convex programming duality theory gives the cost-savings function as

$$\Phi^t(0,0,\ldots,0) - \Phi^t(r_1,r_2,\ldots,r_M) =$$

$$\underset{\substack{\mu \geq 0 \\ \geq 0}}{\text{minimum}} \{\Phi^t(0) - \gamma^t(r_1,r_2,\ldots,r_M;\ \mu,\ \pi_1,\pi_2,\ldots,\pi_M)\}$$

where

$$\gamma^t(r_1,r_2,\ldots,r_M;\ \mu,\pi_1,\pi_2,\ldots,\pi_M) = -\sum_{m=1}^{M} \pi_m^t r_m +$$

$$\min \{c^t(x^t) + f^t(s^t) + \mu^t F^t(d^t;\ \{x^t,s^t,z_1^t,z_2^t,\ldots,z_M^t\}) + \sum_{m=1}^{M} \pi_m^t z_m^t\}$$

$$\text{s.t.} \qquad 0 \leq x^t$$

$$0 \leq s^t \leq s^t$$

$$0 \geq z^t$$

On the supply side of the economy we assume for simplicity that the reserves of each depletable resource R_m, m=1,2,...,M are owned and controlled by individual suppliers. The case of multiple depletable resources will in general involve some arbitrariness on the economic

sector's willingness to pay the individual resource owners. However, as we shall see next, in the situation of a collusive agreement between the suppliers of the multiple depletable resources there are no major departures from the single depletable resource case in Chapter 2.

Suppose the depletable resources' suppliers collude to decide on a supply schedule $(r_m^1, r_m^2, \ldots, r_m^T)$ over a planning horizon T by jointly maximizing the present value of profits resulting from their selling decisions. They jointly solve the *multiple depletable resources suppliers' problem*

$$V_T(R_1, R_2, \ldots, R_M) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta^t(r_1, r_2, \ldots, r_M) - \sum_{m=1}^{M} g_m(\sum_{k=1}^{t-1} r_m^k, r_m^t)\}$$

$$+ \alpha^T \beta^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t, R_2 - \sum_{t=1}^{T} r_2^t, \ldots, R_M - \sum_{t=1}^{T} r_m^t)$$

$$\text{s.t.} \qquad \sum_{t=1}^{T} r_m^t \leq R_m \qquad\qquad m=1,2,\ldots,M \qquad (3.31)$$

$$r_m^t \geq 0 \qquad t=1,2,\ldots,T\ ,\ m=1,2,\ldots,M$$

By the same argument of Chapter 2, we define the multiple depletable resources suppliers' *joint revenue function* as

$$\beta^t(r_1, r_2, \ldots, r_M) = \phi^t(0,0,\ldots,0) - \phi^t(r_1, r_2, \ldots, r_M)$$

and establish as an *equilibrium* condition

"For any non-negative resource levels $r_m^1, r_m^2, \ldots, r_m^T$ $m=1,2,\ldots,M$ satisfying $\sum_{t=1}^{T} r_m^t \leq R_m$, we say that the multiple depletable resources suppliers' and the sectoral problems are in *equilibrium* if these resource levels permit the suppliers to maximize joint profits; namely (3.32)

$$V_T(R_1,R_2,\ldots,R_M) = \sum_{t=1}^{T} \alpha^{t-1} \{\Phi^t(0,0,\ldots,0) - \Phi(r_1^t,r_2^t,\ldots,r_M^t) - $$

$$- \sum_{m=1}^{M} g_m^t (\sum_{k=1}^{t-1} r_m^k, r_m^t)\}$$

$$+ \alpha^T \beta^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t, R_2 - \sum_{t=1}^{T} r_2^t, \ldots, R_M - \sum_{t=1}^{T} r_M^t) \text{ ."}$$

An iterative scheme essentially similar to that of section 2.III can be devised to solve for equilibrium as defined in (3.32). We need only observe that at each iteration, the solution to the multiple depletable resources suppliers' problem (3.31) with the upper-bound approximations $\tilde{\beta}^{t,n}$ substituting for β^t, will result in M supply schedules $(r_m^{1,n}, r_m^{2,n}, \ldots, r_m^{T,n})$, $m=1,2,\ldots,M$. The sectoral problem reoptimization at each iteration will generate for each period t, $t=1,2,\ldots,T$ a vector of bid prices $(\pi_1^t, \pi_2^t, \ldots, \pi_M^t)$ for marginal depletable resource units. These are the

shadow prices associated with constraints (3.29a) or (3.30b).

The proofs of finite and infinite convergence to equilibrium (3.32) of the above described iterative scheme, associated respectively with multiple depletable resources sectoral problems (3.29) and (3.30) are equivalent to those of Theorems 2.1 and 2.2. For this reason we state the two following convergence theorems without proof.

Theorem 3.6: (Finite Convergence) The iterative scheme described above converges to an equilibrium solution after a finite number of iterations between the multiple depletable resources suppliers' problem (3.31) and the LP-sectoral problems (3.29).

Theorem 3.7: (Infinite Convergence) The iterative scheme described above converges to an equilibrium solution between the multiple depletable resources suppliers' problem (3.31) and the convex sectoral problems (3.30) as the number of iterations tends to infinity.

Equilibrium as defined in (3.32) can be identified as a solution to an underlying optimization problem similar to (2.15) and (2.18) in the cases of LP-sectoral problems (3.29) or more general convex sectoral problems (3.30), respectively. Notice that the multiple depletable resources suppliers' problem (3.31) can be solved at each iteration by a sequence of backward recursions. This is because it can be formulated as a dynamic

programming problem, similar to (2.5), but in the M-dimensional space $(S_1, S_2, \ldots, S_M)$.

In the case the suppliers do not collude there is no unique way, except in a special case to be described next, to associate the revenue function of each individual depletable resource supplier with the cost-savings resulting from the availability of each individual resource. This fact can be explained by the joint savings in the sectoral cost of meeting demand, that are not readily attributed to any individual resource. For simplicity we shall discuss the issue in the framework of an economy with two depletable resources and linear programming sectoral problems of the form

$$\begin{aligned} \Phi^t(r_1, r_2) = \min\ & c^t x^t \\ \text{s.t.}\quad & \rho_1^t x^t \leq r_1 \\ & \rho_2^t x^t \leq r_2 \\ & A\, x^t \geq d^t \\ & x^t \geq 0 \end{aligned} \tag{3.33}$$

We shall let $\beta_i^t(r_i, r_j)$, $i=1,2$, $j \neq i$, denote the revenue of the i-th depletable supplier for r_i units of the resource when r_j is the amount supplied of the j-th depletable resource.

The special case mentioned above occurs when the depletable resources do not interact either directly or indirectly, or in other words, when the sectoral problems (3.33) are separable in r_1 and r_2

$$\Phi^t(r_1,r_2) = \min c_1^t x_1^t + c_2^t x_2^t$$

$$\text{s.t.} \quad \rho_1^t x_1^t \leq r_1$$

$$\rho_2^t x_2^t \leq r_2$$

$$A_1 x_1^t \geq d_1^t$$

$$A_2 x_2^t \geq d_2^t$$

$$0 \leq x_1^t \ , \ 0 \leq x_2^t$$

yielding cost functions in each period t, Φ^t, that are additive

$$\Phi^t(r_1,r_2) = \Phi_1^t(r_1) + \Phi_2^t(r_2) \ .$$

For all purposes this economic sector can be regarded as a hybrid of two economic subsectors that do not interact. Consequently, the individual supplier's revenue functions can be defined as

$$\beta_1^t(r_1,r_2) \equiv \Phi_1^t(0) - \Phi_1^t(r_1) \qquad \text{for all } r_2 \geq 0$$

$$\beta_2^t(r_1,r_2) \equiv \Phi_2^t(0) - \Phi_2^t(r_2) \qquad \text{for all } r_1 \geq 0$$

In the case of additive Φ^t the multiple depletable resource equilibrium problem falls back into the basic model described in Chapter 2. Each subsectoral problem and the corresponding depletable resource supplier can iterate to reach equilibrium independently. Example 3.1(i) at the end of this section illustrates a case of a separable cost function.

Ruling out the exceptional case of additivity in Φ^t, we will assume that each depletable resource supplier (i=1,2) determines his supply schedule, $r_i^1, r_i^2, \ldots, r_i^T$, over a planning horizon T when $r_j^1, r_j^2, \ldots, r_j^T$, $(j \neq i)$ is the supply schedule of the alternative resource by maximizing the present value of profits at these levels of the alternative resource. The *i-th depletable resource supplier's problem* is for i=1,2 and $j \neq i$

$$V_T(R_i; r_j^1, r_j^2, \ldots, r_j^T) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta_i^t(r_i^t, r_j^t) - g_i^t(\sum_{k=1}^{t-1} r_i^k, r_i^t)\}$$

$$+ \alpha^T \beta_i^{T+1}(R_i - \sum_{t=1}^{T} r_i^t, R_j - \sum_{t=1}^{T} r_j^t) \qquad (3.34)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r_i^t \leq R_i$$

$$r_i^t \geq 0 \qquad t=1,2,\ldots,T$$

Four different possibilities for defining the suppliers' revenue function, $\beta_i^t(r_i,r_j)$ $(i=1,2,\ j\neq i)$, will be examined. These are:

Case 1:

$$^1\beta_1^t(r_1,r_2) \equiv \Phi^t(0,0) - \Phi^t(r_1,0) \qquad (3.35a)$$

$$^1\beta_2^t(r_1,r_2) \equiv \Phi^t(0,0) - \Phi^t(0,r_2) \qquad (3.35b)$$

Case 2:

$$^2\beta_1^t(r_1,r_2) \equiv \Phi^t(0,0) - \Phi^t(r_1,0) \qquad (3.36a)$$

$$^2\beta_2^t(r_1,r_2) \equiv \Phi^t(r_1,0) - \Phi^t(r_1,r_2) \qquad (3.36b)$$

Case 3:

$$^3\beta_1^t(r_1,r_2) \equiv \Phi^t(0,r_2) - \Phi^t(r_1,r_2) \qquad (3.37a)$$

$$^3\beta_2^t(r_1,r_2) \equiv \Phi^t(0,0) - \Phi^t(0,r_2) \qquad (3.37b)$$

Case 4:

$$^4\beta_1^t(r_1,r_2) \equiv \Phi^t(0,r_2) - \Phi^t(r_1,r_2) \qquad (3.38a)$$

$$ {}^4\beta_2^t(r_1,r_2) \equiv \Phi^t(r_1,0) - \Phi^t(r_1,r_2) \tag{3.38b} $$

Following this discussion, iterative methods to reach the *equilibrium* condition

> "For any non-negative resource $r_1^1, r_1^2, \ldots, r_1^T$ and $r_2^1, r_2^2, \ldots, r_2^T$ satisfying $\sum_{t=1}^{T} r_1^t \leq R_1$ and $\sum_{t=1}^{T} r_2^t \leq R_2$ we say that the economic sector and the two depletable resources suppliers are in *equilibrium* if these resource levels permit the suppliers to maximize their profits; namely

$$ V_T(R_1;r_2^1,r_2^2,\ldots,r_2^T) = \sum_{t=1}^{T} \alpha^{t-1} \{\beta_1^t(r_1^t,r_2^t) - g_1^t(\sum_{k=1}^{t-1} r_1^k, r_1^t)\} + \tag{3.39} $$

$$ + \alpha^T \beta_1^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t,\ R_2 - \sum_{t=1}^{T} r_2^t) $$

$$ V_T(R_2;r_1^1,r_1^2,\ldots,r_1^T) = \sum_{t=1}^{T} \alpha^{t-1} \{\beta_2^t(r_1^t,r_2^t) - g_2^t(\sum_{k=1}^{t-1} r_2^k, r_2^t)\} + $$

$$ + \alpha^T \beta_2^{T+1}(R_1 - \sum_{t=1}^{T} r_1^t,\ R_2 - \sum_{t=1}^{T} r_2^t)." $$

will be explored for each of the above four cases. The properties of the above defined revenue functions (3.35), (3.36), (3.37) and (3.38) are summarized by

Lemma 3.9: The revenue functions $\beta_i^t(r_i,r_j)$ $(i=1,2,\ i\neq j)$ as defined by (3.35), (3.36), (3.37) and (3.38) are

(i) non-decreasing in r_i for fixed r_j

(ii) concave in r_i for fixed r_j

(iii) continuous in r_i and r_j

Proof: Follows trivially from the fact that ϕ^t is assumed non-increasing in r_i and r_j and convex. Furthermore the revenue functions will be continuous if ϕ^t is continuous in r_i and r_j. ||

Lemma 3.10: For cost functions ϕ^t generated by the LP-sectoral problems (3.21), the revenue functions $\beta_i^t(r_i,r_j)$, $(i=1,2,\ j\neq 1)$ as defined by (3.35), (3.36), (3.37) and (3.38) will be piecewise linear in r_i for each fixed r_j.

Proof: This is a direct consequence of the piecewise linearity of ϕ^t as stated in Lemma 3.7.||

Case 1, (3.35), is one in which the economic sector is willing to pay the individual suppliers the cost-savings resulting uniquely from the availability of each individual depletable resource. Any joint economies that may result from the simultaneous availability of both resources are absorbed by the economic sector and are not paid to any of the individual resources suppliers.

Cases 2 and 3 would result from arbitrary priority rules. In Case 2, (3.36), the supplier of the first depletable resource would receive the cost-savings resulting from its early introduction, and the supplier of the second depletable resource the remaining cost-savings. A symmetric observation is valid for Case 3, (3.37). Notice that for Cases 2 and 3,

$$ {}^{2}\beta_1^t(r_1,r_2) + {}^{2}\beta_2^t(r_1,r_2) = \phi^t(0,0) - \phi^t(r_1,r_2) $$

$$ {}^{3}\beta_1^t(r_1,r_2) + {}^{3}\beta_2^t(r_1,r_2) = \phi^t(0,0) - \phi^t(r_1,r_2) $$

or in other words, the sectoral expenditures with both depletable resources would sum the sectoral cost-savings.

Case 4, (3.38), is one in which the economic sector is willing to pay each of the individual resources suppliers, the cost-savings introduced by that resource over and above the availability of the alternative resource. For Cases 1 and 4 we can show

<u>Lemma 3.11</u>: If $\phi^t(0,0) - \phi^t(r_1,0) \; (\gtreqless) \; \phi^t(0,r_2) - \phi^t(r_1,r_2)$ then

$$ {}^{1}\beta_1^t(r_1,r_2) + {}^{1}\beta_2^t(r_1,r_2) \; (\gtreqless) \; \phi^t(0,0) - \phi^t(r_1,r_2) $$

$$ {}^{4}\beta_1^t(r_1,r_2) + {}^{4}\beta_2^t(r_1,r_2) \; (\lesseqgtr) \; \phi^t(0,0) - \phi^t(r_1,r_2) \; . $$

Proof: By observing that

$${}^{1}\beta_1^t(r_1,r_2) + {}^{1}\beta_2^t(r_1,r_2) + {}^{4}\beta_1^t(r_1,r_2) + {}^{4}\beta_2^t(r_1,r_2) =$$

$$= 2[\Phi^t(0,0) - \Phi^t(r_1,r_2)]$$

and, that

$${}^{1}\beta_1^t(r_1,r_2) + {}^{1}\beta_2^t(r_1,r_2) = 2\,\Phi^t(0,0) - \Phi^t(r_1,0) - \Phi^t(0,r_2) =$$

$$= \Phi^t(0,0) - \Phi^t(r_1,0) - \Phi^t(0,r_2) + \Phi^t(r_1,r_2) +$$

$$+ \Phi^t(0,0) - \Phi^t(r_1,r_2)$$

the above result follows. ||

What the above lemma suggests is that under Cases 1 and 4 it is possible for the total sectoral expenditures to strictly exceed the sec-total cost-savings. Of course, as the economic sector finds out that in equilibrium it will be paying more than what the resources save in costs, the sector will be better off by consuming none of the resources. This would work as an incentive for the suppliers of the multiple depletable resources to come to an aggrement and jointly solve the multiple depletable resources suppliers' problem (3.31), as they will be better off. This situation is more clearly seen in the case where both resources can only be used together, that we will explore later in this section.

Apparently one explanation for the resulting behavior of the revenue functions ${}^4\beta_1^t$ and ${}^4\beta_2^t$ relies on a typical case of externalities (see Malinvaud [ ]). Each resource has no value for the economic sector in the absence of its complement. However, the presence of one resource would generate an intrinsic value for its complement that is not being accounted for. We shall now examine some extreme cases to illustrate the results of Lemma 3.11.

We will say that the depletable resource i (i=1,2) is a *complement* to depletable resource j(j≠i) if for every activity in k in (3.33) for which $\rho_{ik} > 0$ we have also $\rho_{jk} > 0$. It is then trivial to verify that for the sectoral problems (3.33)

$$\phi^t(r_i,0) = \phi^t(0,0)$$

for all $r_i \geq 0$. The impact of this situation for the revenue functions (3.35), (3.36), (3.37) and (3.38) is readily seen if we suppose resource 1 is a complement to resource 2. We will then have

$$ {}^1\beta_1^t(r_1,r_2) = 0 \qquad \text{for all } r_1 \geq 0 \text{ and } r_2 \geq 0$$

$$ {}^2\beta_1^t(r_1,r_2) = 0 \qquad \text{for all } r_1 \geq 0 \text{ and } r_2 \geq 0$$

$$ {}^4\beta_2^t(r_1,r_2) = \phi^t(0,0) - \phi^t(r_1,r_2)$$

$$\text{for all } r_1 \geq 0 \text{ and } r_2 \geq 0$$

and consequently under the definitions (3.38), Case 4, the sectoral expenditures with both depletable resources will be greater than or equal to its cost-savings. Example 3.1(ii) illustrates the computation of Φ^t for this type of complementarity.

We say that the resources are *perfect complements* if resource 1 is a complement to resource 2 and vice-versa, in which case we will have

$$\Phi^t(r_1,0) = \Phi^t(0,0) \qquad \text{for } r_1 \geq 0$$

$$\Phi^t(0,r_2) = \Phi^t(0,0) \qquad \text{for } r_2 \geq 0$$

It can be seen that in this situation

$$^1\beta_1^t(r_1,r_2) = 0 \qquad \text{and } {}^1\beta_2^t(r_1,r_2) = 0$$

$$^2\beta_1^t(r_1,r_2) = 0 \qquad \text{and } {}^2\beta_2^t(r_1,r_2) = \Phi^t(0,0) - \Phi^t(r_1,r_2)$$

$$^3\beta_1^t(r_1,r_2) = \Phi^t(0,0) - \Phi^t(r_1,r_2) \text{ and } {}^3\beta_2^t(r_1,r_2) = 0$$

$$^4\beta_1^t(r_1,r_2) = \Phi^t(0,0) - \Phi^t(r_1,r_2) \text{ and } {}^4\beta_2^t(r_1,r_2) = \Phi^t(0,0) - \Phi^t(r_1,r_2)$$

for all $r_1 \geq 0$ and $r_2 \geq 0$. Consequently in Case 4, the sectoral expenditures with both depletable resources will equal twice its cost-savings, while in Case 1, the revenue functions are identically zero. Example 3.1(iii) at the end of the section illustrates a case of perfect complementarity as above defined.

We say that the resources are *perfect substitutes* if there is no activity in (3.33) that utilizes simultaneously both resources and if for every activity k such that $\rho_{1k} > 0$ there exists another activity k* with $\rho_{2k^*} = \kappa\rho_{1k}$ and $a_{k^*} = a_k$ and $c_{k^*} = c_1$. Under these conditions the sectoral problem (3.33) can be rewritten as

$$\phi^t(r_1, r_2) = \min\ c_o^t x_o^t + c_1^t x_1^t + c_1^t x_2^t$$

$$\text{s.t.} \qquad \rho_1^t x_1^t \qquad \leq r_1$$

$$\kappa\rho_1^t x_2^t \leq r_2 \qquad (3.40)$$

$$A_o^t x_o^t + A_1^t x_1^t + A_1^t x_2^t \geq d^t$$

$$0 \leq x_o^t\ ,\ x_1^t\ ,\ x_2^t$$

We can then show

<u>Lemma 3.12:</u> $\phi^t(r_1, r_2) = \phi^t(r_1 + \frac{r_2}{\kappa}\ , 0)$

Proof: Let $\hat{x}_o^t, \hat{x}_1^t$ and $\hat{x}_2$ be optimal for (3.40) then,

$$x_o^t = \hat{x}_o^t \quad \text{and} \quad x_1^t = \hat{x}_1^t + \hat{x}_2^t$$

are feasible for $\Phi^t(r_1 + \frac{r_2}{\kappa}, 0)$ with value $\Phi^t(r_1, r_2)$, yielding

$$\Phi^t(r_1 + \frac{r_2}{\kappa}, 0) \leq \Phi^t(r_1, r_2) \quad .$$

Let $\hat{x}_o^t, \hat{x}_1^t$ be optimal for $\Phi^t(r_1 + \frac{r_2}{\kappa}, 0)$ then

$$x_o^t = \hat{x}_o^t \; , \; x_1^t = \frac{\kappa r_1}{\kappa r_1 + r_2} \hat{x}_1^t \quad \text{and} \quad x_2^t = \frac{r_2}{\kappa r_1 + r_2} \hat{x}_1^t$$

are feasible for $\Phi^t(r_1, r_2)$ with value $\Phi^t(r_1 + \frac{r_2}{\kappa}, 0)$

Consequently

$$\Phi^t(r_1, r_2) \leq \Phi^t(r_1 + \frac{r_2}{\kappa}, 0)$$

which completes the proof. ||

<u>Lemma 3.13:</u> If the resources are perfect substitutes as defined above then

$${}^4\beta_1^t(r_1, r_2) + {}^4\beta_2^t(r_1, r_2) \leq \Phi^t(0,0) - \Phi^t(r_1, r_2) \, .$$

Proof: Since Φ^t is convex and non-increasing

$$\Phi^t(0,0) - \Phi^t(r_1,0) \geq \Phi^t(\frac{r_2}{\kappa}, 0) - \Phi^t(r_1 + \frac{r_2}{\kappa}, 0)$$

From Lemma 3.12 we have

$$\Phi^t(0,0) - \Phi^t(r_1,0) \geq \Phi^t(0,r_2) - \Phi^t(r_1,r_2)$$

and from Lemma 3.11 the above result follows.||

Example 3.1(iv) at the end of this section illustrates a case of perfect substitute resources.

Apparently no a priori conclusions can be drawn when the behavior of the depletable resources (complements or substitutes) varies from one productive activity to another. For an economy with more than two depletable resources it is felt that the newly added complexity would further complicate the above analysis. For this reason it will not be treated here.

For Case 1, (3.35), an iterative scheme to search for equilibrium (3.39) between the sectoral problems (3.29) or (3.30) and the depletable resources suppliers' problems (3.34) can be devised as an immediate extension of the single depletable resource iterative scheme in Chapter 2. Since the revenue function ${}^1\beta_1^t$ and ${}^1\beta_2^t$ do not depend on r_2 and r_1 respectively, the sectoral problem can iterate independently with each depletable resource supplier's problem (3.31), i=1,2, in the absence of the other depletable resource, $r_j^t = 0$, t=1,...,T, j≠i. Convergence to

equilibrium (3.39) under the definitions of Case 1 is guaranteed through Theorems 2.1 and 2.2 respectively.

For Cases 2 and 3 equilibrium can be reached by a two-step iterative scheme. Due to symmetry of the two cases we need examine only Case 2. During the first step the sectoral problems iterate with the supplier of the first depletable resource at $r_2^t = 0$ for $t=1,\dots,T$. Once partial equilibrium between the sectoral problems and the supplier of the first depletable resource is reached, the iterative process stops and we proceed to the second step. During the second step the sectoral problems iterate with the second depletable resource supplier's problem (3.34), at a fixed supply schedule $r_1^t, t=1,\dots,T$, that results at the end of the first step.

Notice that under Cases 1, 2 and 3, the two depletable resources equilibrium (3.39) falls back into two single depletable resource equilibria, for which convergence under the LP-sectoral problems and the more general convex program has been proved in Theorems 2.1 and 2.2 respectively.

Case 4 implies a major departure from the single depletable resource framework as the interdependence of the supplies is recognized by the economic sector, through the definitions of the revenue functions ${}^4\beta_1^t$ and ${}^4\beta_2^t$ in (3.38). The existence of an equilibrium (3.39) in this case can be shown using the fixed point arguments applied to the case of multiple suppliers of a unique depletable resource, described in section 3.V. For revenue functions ${}^4\beta_1^t$ and ${}^4\beta_2^t$, $t=1,2,\dots,T$, that are concave in r_1 and r_2 respectively and continuous in both arguments,

extraction cost functions g_1^t and g_2^t that are convex and continuous in both arguments and salvage functions β_1^{T+1} and β_2^{T+1} concave in S_1 and S_2 respectively and continuous in both arguments, the existence of a Katukani fixed point can be shown as in Theorem 3.5 guaranteeing the existence of equilibrium (3.39).

A doubly iterative scheme, which involves the sequential repetition of the iterative scheme for the single depletable resource case, described in section 2.III, between the sectoral problems and each depletable resource supplier's problem (3.34) is suggested. At step ℓ the first depletable resource supplier's problem (3.34) is solved at a fixed supply schedule of the second depletable resource $(r_2^{1,\ell}, r_2^{2,\ell}, \ldots, r_2^{T,\ell})$, by iterating with the sectoral problems. Once the process converges to an equilibrium in the sence of Chapter 2, it will result in a vector of non-negative resource levels $(r_1^{1,\ell}, r_1^{2,\ell}, \ldots, r_1^{T,\ell})$ satisfying $\sum_{t=1}^{T} r_1^{t,\ell} \leq R_1$. With the first depletable resource supply schedule fixed at those levels, the second depletable resource supplier's problem (3.34) is solved by iterating with the sectoral problems. This will result in a vector of non-negative resource levels $(r_2^{1,\ell+1}, r_2^{2,\ell+1}, \ldots, r_2^{T,\ell+1})$ satisfying $\sum_{t=1}^{T} r_2^{t,\ell+1} \leq R_2$. The process then repeats. This is depicted in Figure 3.9 While the sequence of vectors $(r_1^{t,\ell}, r_2^{t,\ell})$ for t=1,...,T, will have a convergent subsequence because of the compactness implied by the reserves constraints, we shall not pursue here to prove or determine conditions under which the limit of the subsequence is the desired fixed point.

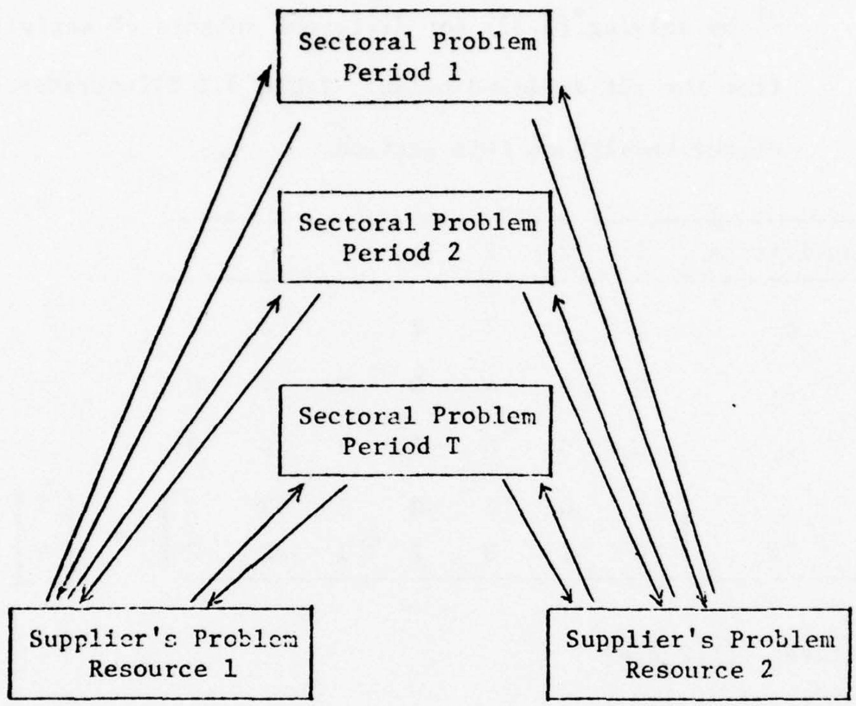

Sectoral Problem
Period 1
Sectoral Problem
Period 2
Sectoral Problem
Period T
Supplier's Problem
Resource 1
Supplier's Problem
Resource 2

Figure 3.9

Example 3.1: In this numerical example we compute the cost function ϕ^t by solving (3.33) for different subsets of activities from the set depicted below. Table 3.1 illustrates some of the results of this section.

Activities	1	2	3	4	5	6	7
c	2	2	2	2	1	2	2
ρ_1	0	0	4	0	2	4	0
ρ_2	0	0	0	2	1	0	2

$$A_2 = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 & 2 \end{bmatrix} \quad d = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

i) Activities: [1,2,3,4]

$$\phi(r_1,r_2) = \min 2x_1 + 2x_2 + 2x_3 + 2x_4$$

$$\begin{aligned} \text{s.t.} \quad & 4x_3 \le r_1 \\ & 2x_4 \le r_2 \\ & x_1 + 2x_3 \ge 6 \\ & x_2 + 2x_4 \ge 4 \\ & 0 \le x_1, x_2, x_3, x_4 \end{aligned}$$

$$\Phi(r_1,r_2) = \begin{cases} 20 - \frac{r_1}{2} - r_2 & 0 \le r_1 \le 12 \quad , \quad 0 \le r_2 \le 4 \\ 14 - r_2 & r_1 \ge 12 \quad , \quad 0 \le r_2 \le 4 \\ 16 - \frac{r_1}{2} & 0 \le r_1 \le 12 \quad , \quad r_2 \ge 4 \\ 10 & r_1 \ge 12 \quad , \quad r_2 \ge 4 \end{cases}$$

This is a case of additive Φ, as can be readily verified.

ii) Activities: [1,2,3,5]

$$\Phi(r_1,r_2) = \min 2x_1 + 2x_2 + 2x_3 + x_5$$

$$\begin{aligned} \text{s.t.} \qquad & 4x_3 + 2x_5 \le r_1 \\ & x_5 \le r_2 \\ & x_1 + 2x_3 + x_5 \ge 6 \\ & x_2 + x_5 \ge 4 \\ & 0 \le x_1,x_2,x_3,x_5 \end{aligned}$$

$$\Phi(r_1,r_2) = \begin{cases} 20 - \frac{3}{2} r_1 & 0 \le r_1 \le 8 \\ 12 - \frac{r_1}{2} & 8 \le r_1 \le 12 \quad \frac{r_1}{2} \le r_2 \\ 6 & 12 \le r_1 \\ 20 - \frac{r_1}{2} - 2r_2 \quad 0 \le r_1 \le 12 , & 0 \le r_2 \le 4 \\ 12 - \frac{r_1}{2} \quad 8 \le r_1 \le 12 , & 4 \le r_2 \quad . \quad \frac{r_1}{2} \ge r_2 \\ 14 - 2r_2 \quad r_1 \ge 12 , & r_2 \le 4 \\ 6 \quad r_1 \ge 12 , & r_2 \ge 4 \end{cases}$$

In this case resource 1 is a complement to resource 2.

iii) Activities: [1,2,5]

$$\Phi(r_1,r_2) = \min 2x_1 + 2x_2 + x_5$$

$$\text{s.t.} \quad \begin{aligned} 2x_5 &\le r_1 \\ x_5 &\le r_2 \\ x_1 \quad + x_5 &\ge 6 \\ x_2 + x_5 &\ge 4 \end{aligned}$$

$$0 \leq x_1, x_2, x_5$$

$$\phi(r_1,r_2) = \begin{cases} 20 - 3r_2 & 0 \leq r_2 \leq 4 & \\ 12 - r_2 & 4 \leq r_2 \leq 6 & r_2 \leq \frac{r_1}{2} \\ 6 & 6 \leq r_2 & \\ 20 - \frac{3r_1}{2} & 0 \leq r_1 \leq 8 & \\ 12 - \frac{r_1}{2} & 8 \leq r_1 \leq 12 & r_2 \geq \frac{r_1}{2} \\ 6 & 12 \leq r_1 & \end{cases}$$

This is a case of perfect complementarity between resources.

iv) Activities: [1,2,6,7]

$$\phi(r_1 r_2) = \min 2x_1 + 2x_2 + 2x_6 + 2x_7$$

$$\begin{aligned} \text{s.t.} \quad & 4x_6 \leq r_1 \\ & 2x_7 \leq r_2 \\ & x_1 + 2x_6 + 2x_7 \geq 6 \\ & x_2 + 2x_6 + 2x_7 \geq 4 \end{aligned}$$

$$0 \le x_1, x_2, x_6, x_7$$

$$\phi(r_1,r_2) = \begin{cases} 20 - \frac{3r_1}{2} - 3r_2 & 0 \le r_1 \le 8 \quad , \quad r_1 + 2r_2 \le 8 \\ 12 - \frac{r_1}{2} - r_2 & 0 \le r_1 \le 8 \ , \ 8 \le r_1 + 2r_2 \le 12 \\ 6 & 0 \le r_1 \le 8 \quad , \quad r_1 + 2r_2 \ge 12 \\ 12 - \frac{r_1}{2} - r_2 & 8 \le r_1 \le 12 \quad , \quad r_1 + 2r_2 \le 12 \\ 6 & 8 \le r_1 \le 12 \quad , \quad r_1 + 2r_2 \ge 12 \\ 6 & r_1 \ge 12 \end{cases}$$

This example is one for which the resources are perfect substitutes.

v) Activities: [1,2,3,4,5]

$$\phi(r_1,r_2) = \min 2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5$$

$$\begin{aligned} \text{s.t.} \quad & 4x_3 \qquad 2x_5 \le r_1 \\ & 2x_4 + x_5 \le r_2 \\ x_1 + \; & 2x_3 \qquad + x_5 \ge 6 \end{aligned}$$

$$x_2 \qquad + 2x_4 \geq 4$$

$$0 \leq x_1, x_2, x_3, x_4, x_5$$

$$\Phi(r_1, r_2) = \begin{cases} 20 - r_1 - r_2 & 0 \leq r_1 \leq 8 \quad , \quad 0 \leq r_2 \leq 4 \\ 16 - r_1 & 0 \leq r_1 \leq 8 \quad , \quad r_2 \geq 4 \quad \frac{r_1}{2} \leq r_2 \\ 12 - \frac{r_1}{2} & 8 \leq r_1 \leq 12 \quad , \quad r_2 \geq 4 \\ 6 & r_1 \geq 12 \quad , \quad r_2 \geq 6 \\ 20 - \frac{r_1}{2} - 2r_2 & 0 \leq r_1 \leq 12 \quad , \quad 0 \leq r_2 \leq 4 \\ 12 - \frac{r_1}{2} & 8 \leq r_1 \leq 12 \quad , \quad 4 \leq r_2 \\ 14 - 2r_2 & r_1 \geq 12 \quad , \quad r_2 \leq 4 \quad \frac{r_1}{2} \geq r_2 \\ 6 & r_1 \geq 12 \quad , \quad r_2 \geq 4 \end{cases}$$

Table 3.1: Comparison of the Alternative Definitions for the Revenue Functions at the Point (4,4)

Cases	Examples i	ii	iii	iv	v
${}^{1}\beta_{1}$	2	2	0	6	2
${}^{1}\beta_{2}$	4	0	0	12	4
${}^{2}\beta_{1}$	2	2	0	6	2
${}^{2}\beta_{2}$	4	4	6	8	6
${}^{3}\beta_{1}$	2	6	6	2	4
${}^{3}\beta_{2}$	4	0	0	12	4
${}^{4}\beta_{1}$	2	6	6	2	4
${}^{4}\beta_{2}$	4	4	6	8	6
$\phi^{t}(0,0) - \phi^{t}(4,4)$	6	6	6	14	8

Chapter 4: Stochastic Considerations on the Sectoral-Supplier Equilibrium

4.I. Introduction

This chapter reconsiders the basic equilibrium model between an economic sector as the sole user of a depletable resource and a unique supplier of this resource under conditions of uncertainty. As mentioned in section 1.I., there are several sources of uncertainty in intertemporal planning with non-augmentable resource stocks. In terms of a one-sector-one-supplier world, the uncertainties directly faced by one economic agent will in general be reflected upon the decisions of the other economic agent. Uncertainties in technology and in the demand for end-use goods are directly faced by the economic sector but will indirectly affect its willingness to pay the resource supplier. On the other hand, extraction and reserves uncertainty are directly faced by the supplier but will indirectly affect the sector's resource availability in each period and consequently its cost of meeting the demand for end-use goods.

According to Tintner [88] there are two extreme behaviors toward uncertainty: active (here-and-now) or passive (wait-and-see). The analysis in the previous chapters is modified in the following sections to deal with some types of uncertainties, according to a prescribed behavior for the two actors involved.

4.II. A Model of Probabilistic Technological Transition

In the framework of Chapter 2, where a unique supplier and unique economic sector iterate to contract over a planning horizon T a depletable

resource supply schedule, the uncertainty in the revenue received by the supplier may have several sources: technology, end-use demands, price of alternative primary supplies, only to cite a few. In this section the analysis in Chapter 2 is modified to take into account the uncertainty in the sectoral technology. By uncertainty in technology, we understand the uncertainty associated with the time period of implementation of new substitute technologies. A formal attempt to model the uncertainty in the timing of technical change in the context of optimal economic growth is due to DasGupta and Heal [14].

We assume that a model of the economic sector is developed to calculate in each period the cost functions

$\Phi^{i,t}(r)$ = minimum cost of meeting demand for the end-use goods under technology i in period t when r is the quantity of depletable resource available to the sector in period t. (4.1)

It is desirable that $\Phi^{i,t}$ be non-increasing and convex. This will be the case if $\Phi^{i,t}$ is computed from a linear program (2.10) or more generally a convex program (2.16) sectoral problem. Further, assuming $\Phi^{i,t}(0) < +\infty$, based on the same arguments of section 2.II, associated with each alternative technology we may define the revenue functions:

$$\beta^{i,t}(r) \equiv \Phi^{i,t}(0) - \Phi^{i,t}(r). \tag{4.2}$$

If the date of the introduction of the alternative technologies is uncer-

tain, the revenue of a supplier receiving the economic sector's cost-savings (4.1) is also uncertain.

For expositional simplicity we shall concentrate here, without loss of generality, on the case of two alternative technological possibilities, i=1,2, denoting the present and the new (future or long-run) technologies respectively. We assume that the transition to the future technology is an irreversible process and that it occurs at the beginning of time period t, not known with certainty but having a probability mass function p_t. We let p_t, t=2,3,...., denote the probability that the transition to the new technology occurs by the start of period t.

We further assume that the depletable resource supplier decides on a supply schedule by maximizing the expected present value of profits. The supplier's problem under sectoral technological uncertainty is more clearly pictured in terms of a dynamic programming formulation.

For this purpose, we define, for $0 \le S \le R$, the functions:

$V_T^{1,t}(S)$ = maximum *expected* present value of profits resulting from selling decisions from period t through T, when S is the stock of depletable resource and the new technology has not been implemented by the start of period t. (4.3)

$V_T^{2,t}(S)$ = maximum present value of profits resulting from selling decisions from period t through T after transition (4.4)

to the new technology, and S is
the stock of depletable resource
by the start of period t.

The difference in definitions (4.3) and (4.4) stem from the fact that once the new technology (i=2) is introduced, all uncertainty is resolved, as it cannot be superceded. This observation is still valid for a long-run technology in the case of a longer sequence of successive substitute technologies.

The functions $V_T^{1,t}$ and $V_T^{2,t}$ satisfy the recursions

$$V_T^{1,t}(S) = \underset{0 \le r \le S}{\text{maximum}} \left\{ \beta^{1,t}(r) - g^t(R - S, r) + \alpha \left\{ \left[\frac{p_{t+1}}{(1 - \sum_{i=2}^{t} p_i)} \right] V_T^{2,t+1}(S - r) + \left\{ 1 - \left[\frac{p_{t+1}}{1 - \sum_{i=2}^{t} p_i} \right] \right\} V_T^{1,t+1}(S - r) \right\} \right\} \tag{4.5}$$

$$t=T, T-1, \ldots, 1$$

$$V_T^{2,t}(S) = \underset{0 \le r \le S}{\text{maximum}} \{ \beta^{2,t}(r) - g^t(R - S, r) + \alpha V_T^{2,t+1}(S - r) \} \tag{4.6}$$

$$t=T, T-1, \ldots, 1$$

where

$$V_T^{i,T+1}(S) = \beta^{i,T+1}(S) \qquad i=1,2, \text{ for all } S.$$

The supplier's problem under sectoral technological uncertainty is solved by computing

$$V_T^{1,1}(R). \tag{4.7}$$

From (4.5) the functions $V_T^{1,t}$ are calculated recursively from a weighted average of $V_T^{1,t+1}$ and $V_T^{2,t+1}$. The weights are the conditional probabilities of transition to the future technology (i=2) by the start of period t+1, given that it has not been implemented by start of period t.

If the revenue functions $\beta^{1,t}$ and $\beta^{2,t}$ are concave, the extraction cost function g^t is convex and the salvage functions are concave, it can be shown recursively as in Lemma 2.3 that the functions $V^{1,t+1}$ and $V^{2,t+1}$ are concave. Hence the computation of (4.5) and (4.6) involves the maximization of a concave objective. Furthermore, for piecewise linear extraction cost functions g^t, salvage functions and revenue functions $\beta^{1,t}$ and $\beta^{2,t}$ which will result either from the LP-sectoral problems (2.10) or as grid linearizations of convex sectoral problems (2.16), the functions $V^{1,t}$ and $V^{2,t}$ in (4.5) and (4.6) can be generated by two equivalent linear programs similar to (2.24).

The supplier's problem under sectoral technological uncertainty (4.7) will result in at most T-1 distinct supply schedules contingent upon the time period of introduction of the new technology. It can be solved iteratively by introducing simultaneously in each period two upper-bound approximations (2.13) or (2.17) to the revenue functions $\beta^{1,t}$ and $\beta^{2,t}$. At each iteration the supplier's problem under sectoral technological uncertainty can be solved using those approximations yielding at most T-1 distinct supply schedules. These are sent to the respective sectoral problems in each period to be priced out. Either a solution to (4.7) with (4.2) is found or a tighter approximation is obtained to at least one of $\beta^{1,t}$ or $\beta^{2,t}$ in some period t. Figure 4.1 depicts the modified iterative

process.

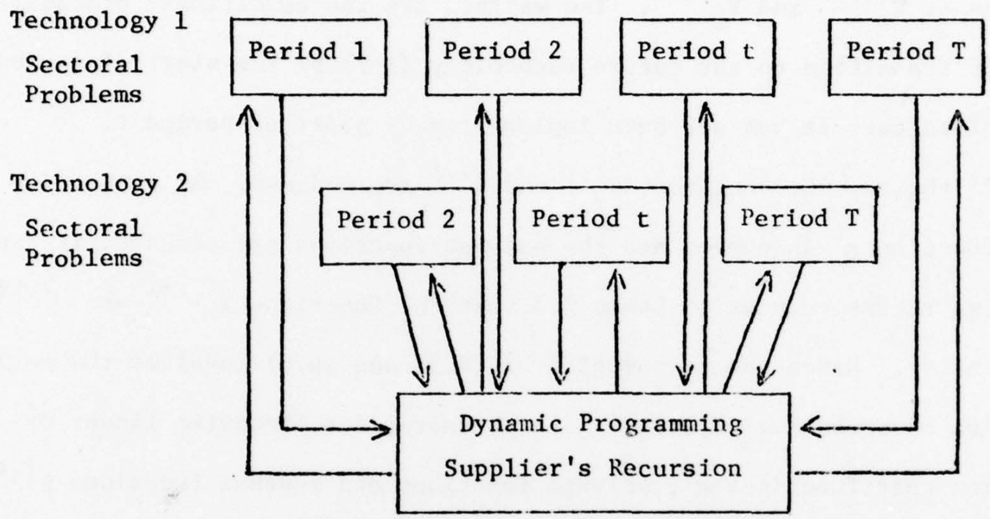


Figure 4.1

We shall now treat a special case which results when we consider a long-run technology for which

$$\phi^{2,t}(0) = \phi^{2,t}(r) \quad \text{for all } r \geq 0.$$

In other words, under the new technology, the depletable resource is no longer a source of cost-savings for the economic sector. The rationale for this situation would be the discovery of a costless resource substitute. As a result of definition (4.2), the supplier's revenue under the sectoral future technology vanishes, giving

$$\beta^{2,t}(r) = 0 \quad \text{for all } r \geq 0 \quad \text{and } t=2,3,\ldots,T.$$

Consequently,

$$V^{2,t}(S) = 0 \quad \text{for all } 0 \leq S \leq R \quad \text{and } t=2.3,\ldots,T.$$

The recursion (4.5) then simplifies to

$$V_T^{1,t}(S) = \underset{0 \le r \le S}{\text{maximum}} \left\{ \beta^{1,t}(r) - g^t(R - S, r) + \alpha \left[1 - \frac{p_{t+1}}{1 - \sum_{i=1}^{t} p_i} \right] V_T^{1,t+1}(S - r) \right\}$$

which are analogous to the recursions (2.5) for the supplier's problem (2.1) if we consider

$$\alpha(t) = \alpha \left[1 - \frac{p_{t+1}}{1 - \sum_{i=2}^{t} p_i} \right] \le \alpha \qquad t=1,2,\ldots,T$$

as an effective discount factor smaller than the riskless discount factor α and possibly non-uniform because of time-dependence. When the transition probabilities follow a geometric distribution with:

$$p_t = \theta^{t-2}(1 - \theta) \qquad t=2,3,\ldots.$$

with $\theta < 1$, we have for the effective discount factor

$$\alpha(t) = \alpha\theta \qquad t=1,2,\ldots,T.$$

Similar conclusions were derived by Heal [38] considering uncertain planning horizons. The observation that an uncertain transition to a long-run technology that makes depletable resource worthless for the economic sector is in effect creating an uncertain relevant planning horizon for sup-

ply decisions, relates the two apparently distinct situations.

This model of probabilistic technological transition could be extended in a number of ways. Only a slight modification of the dynamic programming recursions would be needed to cover the situation in which the probability distribution of the time period of technological transition depends on the stock level of the depletable resource reserves, the state variable S. This would require simply introducing probabilities functionally dependent on resource stocks

$$p_t(S) \qquad t=2,3,\ldots,T$$

in (4.5). Functional forms for p_t could be possibly derived from an R&D model. A step in this direction is the consideration of research and development in the context of resource-constrained optimal economic growth by DasGupta et al. [15]. The question of modelling diffusion in technological transition versus the jump process, considered here, also suggests a potential extension of the above formulation.

4.III. Uncertainty in End-Use Demands

So far we have assumed that the economic sector's cost function Φ^t is calculated under deterministic conditions. In other words, all parameters involved in its computation are supposedly known with certainty over the entire supplier's planning horizon. The perfect foresight assumption is more likely to be violated the longer the supplier's planning horizon. In this section we analyze alternative specifications for the supplier's revenue functions, taking into account the uncertainty in the demand for the sector's end-use goods.

The situation of uncertain demands departs from the original scheme because for random demands, constraints (2.10a) and (2.16b) in the sectoral problems are non-operational. We assume that the economic sector must contract over a planning horizon T with the supplier a depletable resource schedule $r^1, r^2, \ldots, r^T$ *here and now*. The supplier is not willing to let the sector *wait and see* the uncertainty in demand resolve to determine his revenue. This is the framework of long-term contracting with no recontracting clauses. For this reason the sector needs to determine the level of its operations through a two-stage scheme. A first-stage would determine the level of operation of all activities involving a positive depletable resource usage. A second-stage would then establish decision rules contingent upon the first-stage decision, indicating the level of operations of activities that do not utilize any of the depletable resource for each and every possible outcome of demand. The first development of this scheme is Dantzig's [13] two-stage linear model. We assume that a two-stage model of the economic sector is developed to compute in each period

$T^t(r)$ = minimum expected *here and now* cost of meeting the demand for end-use goods in period t when r is quantity of the depletable resource available to the sector in period t.

It is desirable as before that T^t be non-decreasing and concave. We further require it satisfy $T^t(0) < +\infty$.

For convenience we concentrate here on two-stage linear programming sectoral problems and discrete distributions for the end-use demands. The same two-stage approach could be carried for nonlinear models and continuous random variables as shown by Ziemba [96]. For continuous random demands, the two-stage linear model proposed below will fall into the category of continuous linear programming problems (see Grinold [33],[34].

We consider a finite number Q^t of possible values for the end-use demands in period t. Denote these occurrences by d_q^t and the associated probabilities by p_q^t, $q=1,2,\ldots,Q^t$. Partition the vector of resource utilization such that

$$\rho = [\rho_1, 0] \quad \text{with} \quad \rho_1 > 0$$

and let

$$x' = [x_1', x_2'],\ A_1 = [A_{11}, A_{12}],\ A_2 = [A_{21}, A_{22}] \quad \text{and} \quad c = [c_1, c_2]$$

correspondingly.

The *two-stage-LP-sectoral problem* is given by

$$T^t(r) = \min\ c_1^t x_1^t + f^t s_1^t + \sum_{q=1}^{Q^t} p_q^t [c_2^t x_{2q}^t + f^t s_{2q}^t]$$

$$\left.\begin{array}{lll}
\text{s.t.} & \rho_1^t x_1^t \leq r & (4.8a) \\
& A_{11}^t x_1^t - s_1^t \leq 0 & (4.8b) \\
& A_{21}^t x_1^t = z^t & (4.8c) \\
& 0 \leq x_1^t \quad 0 \leq s_1^t \leq s^t & (4.8d) \\
& 0 \leq z^t \leq d^t & (4.8e)
\end{array}\right\} \text{(first-stage)}$$

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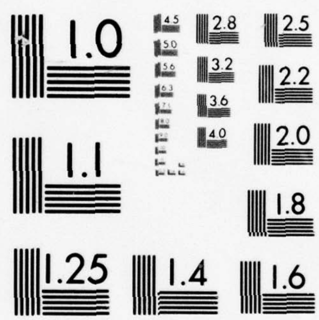

MICROCOPY RESOLUTION TEST CHART
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$$\left.\begin{array}{lll} A_{12}^t x_{2q}^t - s_{2q}^t \le 0 & q=1,2,\ldots,Q^t & (4.8f) \\ A_{22}^t x_{2q}^t \ge d_q^t - z^t & q=1,2,\ldots,Q^t & (4.8g) \\ 0 \le x_{2q}^t \quad 0 \le s_{2q}^t \le s^t - s_1^t & q=1,2,\ldots,Q^t & (4.8h) \end{array}\right\} \text{(second-stage)}$$

where z^t stands for the demands met by first-stage variables restricted by (4.8e) to a lower bound d^t on the possible occurrences of the end-use demands ($d^t \le d_q^t$, $q=1,2,\ldots,Q^t$).

The results of Lemma 2.1 are valid for the linear program (4.8). For this reason we state without proof

<u>Lemma 4.1</u>: T^t given by (4.8) is

(i) non-decreasing and convex;

(ii) piecewise linear with a finite number of segments.

For comparative purposes, we let for $q=1,2,\ldots,Q^t$

$$\Phi_q^t(r) = \min c^t x_q^t + f^t s_q^t$$

$$\rho^t x_q^t \le r \qquad (4.9a)$$

$$A_1^t x_q^t - s^t \le 0 \qquad (4.9b)$$

$$A_2^t x_q^t \ge d_q^t \qquad (4.9c)$$

$$0 \le x_q^t \quad 0 \le s_q^t \le s^t \qquad (4.9d)$$

denote the sector's cost of meeting demand when r units of the depletable resource are available for every possible occurrence of the end-use

demands d_q^t. If the economic sector could *wait and see*, its expected cost of meeting demand would be given by

$$\sum_{q=1}^{Q^t} p_q^t \phi_q^t(r).$$

We can show then:

<u>Lemma 4.2</u>: $$T^t(0) = \sum_{q=1}^{Q^t} p_q^t \phi_q^t(0) \quad \text{and,}$$

$$T^t(r) \geq \sum_{q=1}^{Q^t} p_q^t \phi_q^t(r) \quad \text{for all } r \geq 0.$$

Proof: By letting $r=0$ in the two-stage-LP-sectoral problem (4.8), we can show by (4.8a) that $x_1^t = 0$. Consequently, $s_1^t = 0$ and $z^t = 0$. Therefore only the second-stage remains and problem (4.8) is separable into Q^t problems (4.9), determining $\phi_q^t(0)$, $t=1,2,\ldots,T$. Hence

$$T^t(0) = \sum_{q=1}^{Q^t} p_q \phi_q^t(0).$$

For $r \geq 0$, consider the problem

$$\sum_{q=1}^{Q^t} p_q^t \phi_q^t(r) = \min \sum_{q=1}^{Q^t} p_q^t [c^t x_q^t + f^t s_q^t] \tag{4.10}$$

$$\text{s.t. (4.9a), (4.9b), (4.9c) and (4.9d)} \qquad q=1,2,\ldots,Q^t$$

It is trivial to observe that (4.8) is a more constrained version of (4.10), namely requiring that

$$x_{1q}^t = x_1^t \qquad q=1,2,\ldots,Q^t$$

and including (4.8c). From this observation we conclude that

$$T^t(r) \geq \sum_{q=1}^{Q^t} p_q^t \phi_q^t(r)$$

which completes the proof.||

The difference

$$T^t(r) - \sum_{q=1}^{Q^t} p_q^t \phi_q^t(r)$$

may be called the *cost of uncertainty* in period t. This quantity represents the maximum amount the economic sector would be willing to pay for a perfect prediction of the end-use demands, in period t, when r units of the depletable resource are available.

If we further assume that the economic sector is willing to pay *here and now* for r units of the resource at most its expected here-and-now cost-savings, $T^t(0) - T^t(r)$, and define the supplier's revenue function as

$$\beta^t(r) \equiv T^t(0) - T^t(r), \tag{4.11}$$

we have by Lemma 4.2

$$\beta^t(r) \leq \sum_{q=1}^{Q^t} p_q^t[\phi_q^t(0) - \phi_q^t(r)] \qquad \text{for } r \geq 0.$$

Hence the cost of uncertainty is being passed to the supplier as a penalty for his impatience in terms of a smaller revenue function. Had the supplier the patience to let the economic sector wait and see, he could receive $\phi_q^t(0) - \phi_q^t(r)$ with probability p_q^t, $q=1,2,\ldots,Q^t$.

An equilibrium in terms of (4.11) could be defined similarly to (2.9).

It would be reached through the iterative approach described in section 2.III.1. The sectoral problems (4.8), due to the large number of constraints associated with the number of possible occurrences of demand, could be solved at each iteration by column generation schemes as described in Geoffrion [27].

There are alternative approaches to two-stage models to deal with stochastic parameters. Alternative sector's strategies would give alternative specifications for the sectoral problems that would eventually lead to alternative cost-savings functions. The economic sector could instead determine a cost of meeting demand under the most stringent conditions, in this case, by solving (4.9) only for the smallest possible occurrence of demand. Still the economic sector could solve the deterministic LP-sectoral problems (2.10) using for the demand that must be satisfied the expected demands $\sum_{q=1}^{Q^t} p_q^t d_q^t$. The relationship among these alternative strategies as well as further suggestions are found in Wagner [89].

4.IV. Uncertainty in Reserves and Markovian Decision Theory

In this section we examine the case of a stochastic supplier's problem falling into an important class of stochastic dynamic programming problems called Markovian decision models (see Howard [45]; Shapiro [78]). In Markovian decision models, at the start of each decision period, the state of the process is observed and as a result a decision is selected from a non-empty decision set with the objective to maximize total expected discounted profits over a finite or infinite planning horizon.

We assume here that the stock of depletable resource held by the

supplier R is uncertain, having a known probability distribution function F_R. This assumption makes the supplier's problem (2.1) non-operational as the reserves constraint is not well defined. *Here-and-now* approaches to solve the resulting stochastic supplier's problem are manifold. However these would in general involve the choice of arbitrary loss functions and allowable probability levels of default for chance constraints (see Sengupta [76]). For this reason we concentrate our modelling effort in the case of reserves uncertainty on the *wait-and-see* approach of Markovian Decision Theory.

By the start of each decision period the remaining stock of the depletable resource, S, is non-observable and therefore does not qualify to define the state space for the Markovian decision supplier's problem. This is, in contrast with the deterministic supplier's problem, dynamic programming recursions (2.4) in section 2.II. For the supplier of a depletable resource we define the *state space*

$$\Xi = E \cup \{*\}$$

where E denotes the space of non-depleted states of the cumulative amount extracted and $\{*\}$ the state of final exhaustion. The state of final exhaustion is a random absorption state, that when reached, the decision process is terminated.

As a result of extraction uncertainty, the amount of resource sold in each decision period is uncertain. At the start of each decision period the supplier after observing the state $E \in \Xi$ decides on a *target extraction*, r, from the *action space* $\mathcal{D}(E)$. Given this extraction decision, the

amount actually realized may deviate from the target extraction decision. We let r denote the *actual extraction* or the amount actually obtained and sold by the supplier de facto. Consequently the immediate profits realized by the resource owner in each period are random, given by

$$\Pi^t(E,r,r) = \beta^t(r) - g^t(E,r,r).$$

Dependence of profits on both the target and the realization of the extraction decision are granted by allowing extraction costs to depend on both variables. This would be the case when extraction costs include both setup and variable costs components.

The system then makes a transition to a new state, $Y \in \Xi$. The supplier at the start of the next decision period will either observe that reserves have been exhausted, in which case

$$Y = *$$

or else the new state belongs to the space $\mathcal{E}$, and

$$Y = E + r.$$

Probability distributions for r and Y can be determined for each decision period from subjective probability distributions for the remaining stock of the depletable resource and for the actual extraction given an extraction decision, conditional on the remaining resource stock.

Let $F_S(S;E)$ for $E \in \mathcal{E}$ denote the supplier's subjective probability distribution function for the resource stock of the depletable resource when E units have already been extracted satisfying

$$F_R(R) = F_S(R;0).$$

Let $F_{\tilde{r}|S}(\tilde{r}|S;r)$ denote the supplier's subjective probability distribution function for the actual amount extracted conditional on the remaining stock of the depletable resource when the target extraction decision is r. When extraction uncertainty is uniquely attributed to uncertainty in the remaining stock of the depletable resource

$$\tilde{r} = \text{minimum } \{r, S\}$$

and $F_{\tilde{r}|S}$ takes the form

$$F_{\tilde{r}|S} = \begin{cases} \left.\begin{cases} 0 & \text{for } \tilde{r} \le r \\ 1 & \text{for } \tilde{r} \ge r \end{cases}\right\} & \text{if } S \ge r \\ \left.\begin{cases} 0 & \text{for } \tilde{r} \le S \\ 1 & \text{for } \tilde{r} \ge S \end{cases}\right\} & \text{if } S \le r \end{cases}$$

From the above definition a joint probability distribution function for $\tilde{r}$ and $S, F_{\tilde{r},S}$ and a marginal probability distribution function for $\tilde{r}, F_{\tilde{r}}$, can be derived by

$$dF_{\tilde{r},S}(\tilde{r},S;E,r) = dF_{\tilde{r}|S}(\tilde{r}|S;r)dF_S(S;E)$$

and

$$dF_{\tilde{r}}(\tilde{r};E,r) = \int_S dF_{\tilde{r},S}(\tilde{r},S;E,r).$$

In order to derive the probability distribution of the new state Y we need observe that the probability of exhausting the resource stock when the state is E and the target is r is given by

$$\text{Prob }\{Y = *;E,r\} = \int_S \text{Prob }\{\tilde{r} = S;r\}\ dF_S(S;E) \tag{4.12}$$

where

$$\text{Prob }\{\tilde{r} = S;r\} = F_{\tilde{r}|S}(S + |S;r) - F_{\tilde{r}|S}(S - |S;r)$$

which vanishes if $F_{\pi|S}$ is continuous at $\pi = S$. A probability distribution function for $Y \varepsilon \Xi$, F_Y is then given by (4.12) and for $Y \varepsilon F$ by

$$dF_Y(Y;E,r) = \frac{d}{dY} \int_{\pi=0}^{\pi=Y-E} dF_{\pi}(\pi;E,r).$$

For $E \varepsilon \Xi$ define the functions:

$V_T^t(E)$ = maximum expected present value of profits from selling decisions from period t through T when E units of the resource have been extracted by the start of period t.

The functions V^t satisfy the recursions

$$V_T^t(E) = \underset{r\varepsilon\mathcal{D}(E)}{\text{maximum}} \{\bar{\Pi}^t(E,r) + \alpha \int_{Y\varepsilon\Xi} V_T^{t+1}(Y)dF_Y(Y;E,r)\} \qquad (4.13)$$

$$t=T,T-1,\ldots,1$$

with

$$V_T^t(*) = 0 \qquad t=1,2,\ldots,T,$$

and

$$V_T^{T+1}(E) = \int_S \beta^{T+1}(S)dF_S(S;E)$$

where

$$\bar{\Pi}^t(E,r) = \int \Pi^t(E,\pi,r)dF_{\pi}(\pi;E,r).$$

The finite-horizon *Markovian decision supplier's problem* is solved by computing

$$V_T^1(0). \qquad (4.14)$$

For a supplier receiving as revenue the economic sector's savings in costs

$$\beta^t(\lambda) \equiv \Phi^t(0) - \Phi^t(\lambda) \tag{4.15}$$

the recursions (4.13) can be rewritten as:

$$V_T^t(E) = \underset{r\varepsilon\mathcal{D}(E)}{\text{maximum}} \{ \int [\Phi^t(0) - \Phi^t(\lambda) - g^t(E,\lambda,r)]dF_\lambda(\lambda;E,r) +$$

$$\alpha \int_{Y\varepsilon\Xi} V_T^{t+1}(Y)dF_Y(Y;E,r)\}$$

with Φ^t being derived from a sectoral problem, possibly (2.10) or (2.16). Hence the functions $V_T^t(E)$ are of the form

$$V_T^t(E) = \underset{r\varepsilon\mathcal{D}(E)}{\text{maximum}} \mathcal{E}\{[\ldots \underset{x,s}{\text{minimum}} \ldots]\} \tag{4.16}$$

where $\mathcal{E}$ denotes the expectational operator. This formulation assumes a passive or wait-and-see behavior of the economic sector with respect to the depletable resource supply. Namely the sector will only set the level of operations of its activities and the purchase of alternative primary supplies after the occurrence λ of the target extraction decision r is observed.

For convenience in the discussion below we will assume that the possible outcomes of the extraction decisions are bounded from above and below. We let $\underline{\lambda}(E,r)$ and $\bar{\lambda}(E,r)$ denote respectively lower and upper-bounds on the actual extraction when the state of cumulative extraction is E and the target extraction is set at r. Assuming that an optimal solution to (4.13) exists and denoting it by $\hat{r}^t(E)$, t=1,2,...,T, we notice that if an

approximation to the revenue functions (4.15), $\tilde{\beta}^t$, is being used instead, then

$$V^t(E) = \tilde{V}^t(E)$$

if

$$\tilde{\beta}^t(\lambda) = \phi^t(0) - \phi^t(\lambda) \qquad \text{for all } \underline{\lambda}(E,\hat{r}) \leq \lambda < \bar{\lambda}(E,\hat{r}).$$

Hence we may solve the Markovian decision supplier's problem (4.14) by using upper-bound approximations for the revenue functions. These can be sequentially improved by the addition of new linear segments.

An iterative process can be carried analogously to section 2.III. At iteration n the Markovian decision supplier's problem can be solved using those approximations. It will result in a first period decision r^1, and a sequence of decision rules $r^t(E)$ for $t=2,\ldots,T$. Ex-ante, r^t, $t=2,\ldots,T$ is not known because the state of cumulative extraction E at the start of each period t is random. However since

$$E^t = \sum_{j=1}^{t-1} \lambda^j$$

a probability distribution for E^t and consequently for r^t may be derived. Assuming that this will result in bounded distributions for λ^t, the supplier can announce to the economic sector the sequence

$$\{[\underline{\lambda}^1, \bar{\lambda}^1], [\underline{\lambda}^2, \bar{\lambda}^2], \ldots, [\underline{\lambda}^T, \bar{\lambda}^T]\}.$$

The sector reoptimizes the sectoral problems parametrically at those levels to determine its costs $\phi^t(\lambda)$ and shadow prices $\pi^{t,n+1}$ for all $\underline{\lambda}^{t,n} \leq \lambda \leq$

$\bar{r}^{t,n}$, and $t=1,2,\ldots,T$. Then either

$$\tilde{\beta}^{t,n}(r) = \Phi^t(0) - \Phi^t(r)$$

for $\underline{r}^{t,n} \leq r \leq \bar{r}^{t,n}$, in which case a solution to (4.14) has been found or

$$\tilde{\beta}^{t,n}(r) > \Phi^t(0) - \Phi^t(r)$$

for some $\underline{r}^{t,n} \leq r \leq \bar{r}^{t,n}$ and some $t=1,2,\ldots,T$. In the latter case a new linear segment

$$\Phi^t(0) - \Phi^t(r^{t,n}) + \pi^{t,n+1}(r - r^{t,n})$$

is added to the approximations. The Markovian decision supplier's problem (4.14) is then reoptimized using the resulting new approximation $\tilde{\beta}^{t,n+1}$. Figure 4.2 below depicts the generation of the linear segments in the interval $[\underline{r},\bar{r}]$.

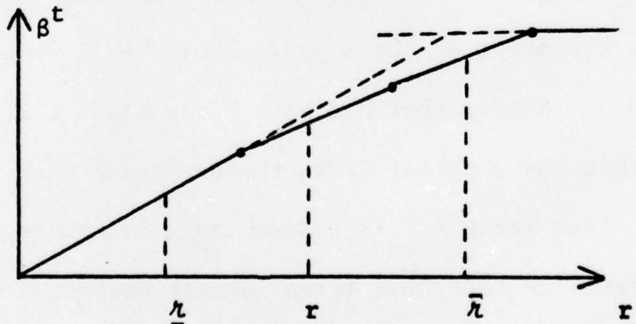


Figure 4.2

We present here an illustrative application of the Markovian decision supplier's problem in discrete action and state spaces. The supplier's

revenue functions are piecewise linear, concave and non-decreasing as generated by the LP-sectoral problems (2.10).

Example 4.1. In each period the target extraction r is selected from the discrete decision set

$$\mathcal{D}(E) = \{0,1,2,3\}$$

for all $E \in \Xi$. From the probability distributions of the stock left, $p(S;E)$, and the conditional probability distribution for the actual amount extracted, $p(\lambda|S;r)$ as given in Tbls. 4.1 and 4.2 respectively, it is trivial to verify that the state space is

$$\Xi = \{0,1,2,3,4,5,*\}.$$

The probability distribution in Table 4.1 is the simplest form of an adaptive distribution for stocks left. It is, in fact, the distribution for reserves R at the start of the planning horizon ($E = 0$) conditional on the event $R - E > 0$. Notice that $p(E;E) = 0$ for all $E \in \Xi$, ruling out the possibility that the supplier is in the depletion state * without knowledge of it. From Table 4.2 we notice that, had there been enough reserves of the resource left, the actual amount extracted would be the result of a 50-50 chance Bernouilli trial at $\lambda = r-1$ and $\lambda = r$. Otherwise the amount of resource left limits the maximum amount that can be realized from a decision to extract r units. The probability distributions for the actual supply, $p(\lambda;E,r)$ and for the resulting next state, $p(Y;E,r)$ are derived as depicted in Tables 4.3 and 4.4 respectively.

We further consider in this numerical example the supplier's planning

Table 4.1. Probability Distribution of Resource Reserves - $p(S;E)$

$E \diagdown S$	3-E	4-E	5-E	6-E
[0,3)	1/4	1/4	1/4	1/4
[3,4)	0	1/3	1/3	1/3
[4,5)	0	0	1/2	1/2
[5,6)	0	0	0	1

Table 4.2. Conditional Probability Distribution of the Actual Supply - $p(\mathcal{r}|S;r)$

$S \diagdown \mathcal{r}$	r-1	r	S
[0,r-1)	0	0	1
[r-1,r)	1/2	0	1/2
[r,6)	1/2	1/2	0

Table 4.3. Probability Distribution of Actual Extraction - $p(\mathcal{r};E,r)$

$p(\mathcal{r};E,0)$

E\\$\mathcal{r}$	0	1	2	3
0	1	0	0	0
1	1	0	0	0
2	1	0	0	0
3	1	0	0	0
4	1	0	0	0
5	1	0	0	0
*	1	0	0	0

$p(\mathcal{r};E,1)$

E\\$\mathcal{r}$	0	1	2	3
0	1/2	1/2	0	0
1	1/2	1/2	0	0
2	1/2	1/2	0	0
3	1/2	1/2	0	0
4	1/2	1/2	0	0
5	1/2	1/2	0	0
*	1	0	0	0

$p(\mathcal{r};E,2)$

E\\$\mathcal{r}$	0	1	2	3
0	0	1/2	1/2	0
1	0	1/2	1/2	0
2	0	5/8	3/8	0
3	0	2/3	1/3	0
4	0	3/4	1/4	0
5	0	1	0	0
*	1	0	0	0

$p(\mathcal{r};E,3)$

E\\$\mathcal{r}$	0	1	2	3
0	0	0	1/2	1/2
1	0	0	5/8	3/8
2	0	2/8	4/8	2/8
3	0	2/6	3/6	1/6
4	0	2/4	2/4	0
5	0	1	0	0
*	1	0	0	0

Table 4.4. Probability Distribution of Next State - $p(Y;E,r)$

$p(Y;E,0)$

E\Y	0	1	2	3	4	5	*
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	0	1	0	0	0	0
3	0	0	0	1	0	0	0
4	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0
*	0	0	0	0	0	0	1

$p(Y;E,1)$

E\Y	0	1	2	3	4	5	*
0	1/2	1/2	0	0	0	0	0
1	0	1/2	1/2	0	0	0	0
2	0	0	1/2	3/8	0	0	1/8
3	0	0	0	1/2	2/6	0	1/6
4	0	0	0	0	1/2	1/4	1/4
5	0	0	0	0	0	1/2	1/2
*	0	0	0	0	0	0	1

$p(Y;E,2)$

E\Y	0	1	2	3	4	5	*
0	0	1/2	1/2	0	0	0	0
1	0	0	1/2	3/8	0	0	1/8
2	0	0	0	3/8	2/8	0	3/8
3	0	0	0	0	1/3	1/6	1/2
4	0	0	0	0	0	1/4	3/4
5	0	0	0	0	0	0	1
*	0	0	0	0	0	0	1

$p(Y;E,3)$

E\Y	0	1	2	3	4	5	*
0	0	0	1/2	3/8	0	0	1/8
1	0	0	0	2/8	1/8	0	5/8
2	0	0	0	0	2/8	1/8	5/8
3	0	0	0	0	0	1/6	5/6
4	0	0	0	0	0	0	1
5	0	0	0	0	0	0	1
*	0	0	0	0	0	0	1

horizon to be two periods long (T = 2) and zero salvage value ($\beta^3(S) = 0$ for all S). The revenue functions β^1 and β^2 are depicted in Figure 4.3. The sector shadow prices are summarized in Table 4.5. Extraction costs are, for simplicity, assumed to depend uniquely in the cumulative past extractions E and the target extraction r as given by

$$g^t(E,r) = \begin{cases} r & E \le 2 \quad r \le 2 - E \\ 2 - E + 2(r - 2 + E) & E \le 2 \quad r \ge 2 - E \\ 2r & E \ge 2 \end{cases}$$

for t=1,2.

For a discount factor, $\alpha = .9$, the Markovian decision supplier's problem is solved after two iterations with the sectoral problems as shown in Tbls. 4.6 and 4.7. The optimal solution is to set

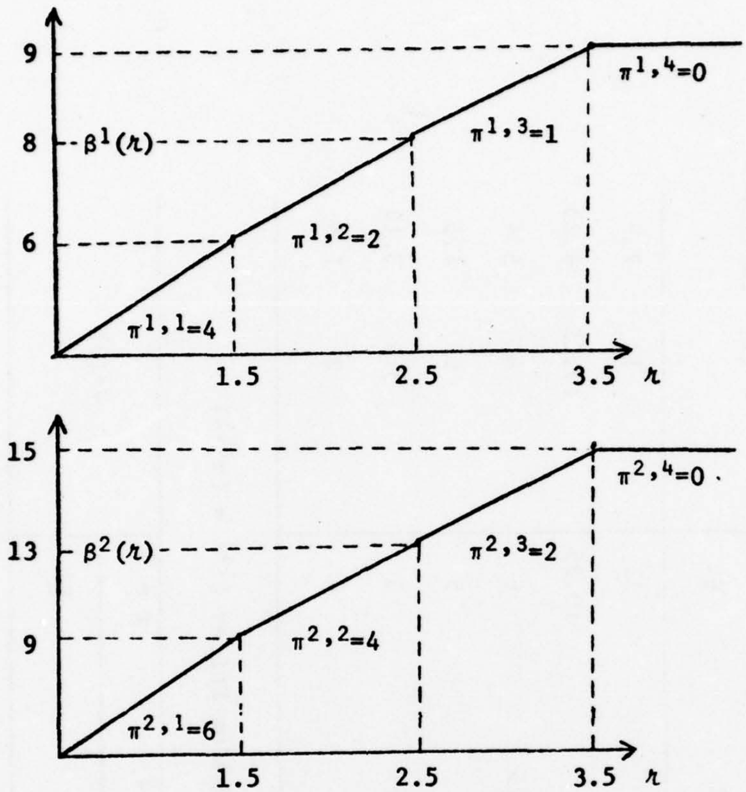
$$\hat{r}^1 = 2$$

$$\hat{r}^2 = 3$$

no matter what state is reached as a result of the first period decision. This is shown in a decision tree format in Figure 4.4. The probability distribution of the state of the system at the end of periods 1 and 2 is presented in Table 4.8.

4.V. A Discussion on the Sector's Active Behavior

The development in the previous section assumed that the economic sector would act passively in each period with respect to the depletable resource supply. In other words the sector would *wait and see* the uncer-



Revenue Functions: β^1 and β^2

Figure 4.3

Table 4.5: Sectoral Shadow Prices - Example 4.1

Shadow Prices - Period 1				Shadow Prices - Period 2			
Range min	Range max	Shadow Prices		Range min	Range max	Shadow Prices	
0	1.5	$\pi^{1,1}$	4	0	1.5	$\pi^{2,1}$	0
1.5	2.5	$\pi^{1,2}$	2	1.5	2.5	$\pi^{2,2}$	4
2.5	3.5	$\pi^{1,3}$	1	2.5	3.5	$\pi^{2,3}$	2
3.5	∞	$\pi^{1,4}$	0	3.5	∞	$\pi^{2,4}$	0

Table 4.6: Iteration 1

Period 2: Known Shadow Prices $L^{2,1} = \{\pi^{2,1}\}$

E	r = 0	r = 1	r = 2	r = 3	$V^2(E)$	$.9V^2(E)$
	$\bar{\Pi}^2$	$\bar{\Pi}^2$	$\bar{\Pi}^2$	$\bar{\Pi}^2$		
0	0	2	7	11	11	9.9
1	0	2	7	10.25	10.25	9.23
2	0	1	4.25	6	6	5.4
3	0	1	4	5	5	4.5
4	0	1	3.5	3	3.5	3.15
5	0	1	2	0	2	1.8

Period 1: Known Shadow Prices $L^{1,1} = \{\pi^{1,1}\}$

E	r = 0	r = 1	r = 2	r = 3	$V^1(0)$
	$\bar{\Pi}^1$	$\bar{\Pi}^1$	$\bar{\Pi}^1$	$\bar{\Pi}^1$	
0	0	1	3.5	2.5	11.32

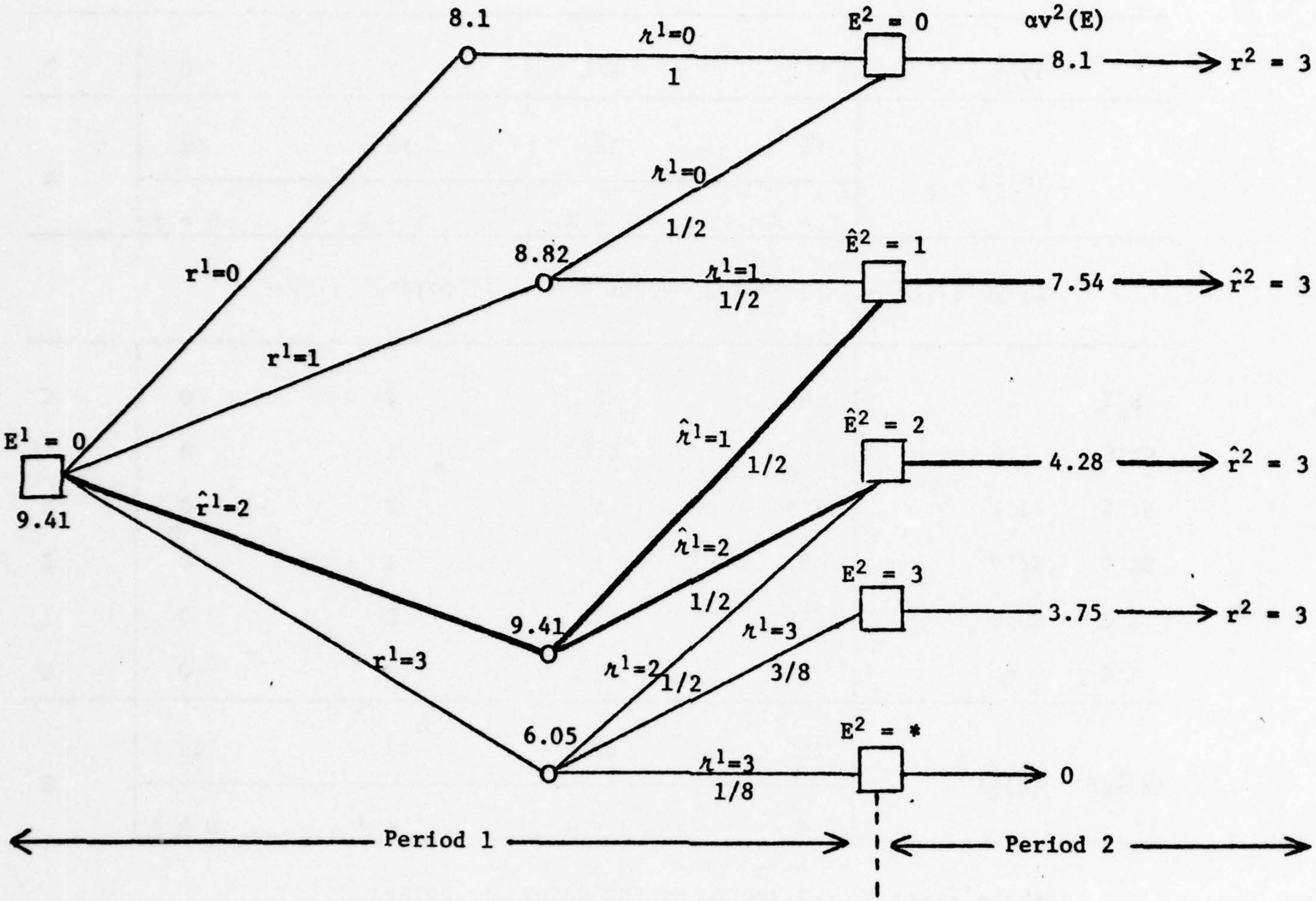
Table 4.7: Iteration 2

Period 2: Known Shadow Prices $L^{2,2} = \{\pi^{2,1}, \pi^{2,3}\}$

E	r = 0	r = 1	r = 2	r = 3	$V_2^2(E)$	$.9V_2^2(E)$
	$\bar{\Pi}^2$	$\bar{\Pi}^2$	$\bar{\Pi}^2$	$\bar{\Pi}^2$		
0	0	2	7	9	9	8.1
1	0	2	7	8.38	8.38	7.54
2	0	1	4	6	4.75	4.28
3	0	1	4	4.17	4.17	3.75
4	0	1	3.5	3	3.5	3.15
5	0	1	2	6	2	1.8

Period 1: Known Shadow Prices $L^{1,2} = \{\pi^{1,1}, \pi^{1,2}\}$

E	r = 0	r = 1	r = 2	r = 3	$V_2^1(0)$
	$\bar{\Pi}^1$	$\bar{\Pi}^1$	$\bar{\Pi}^1$	$\bar{\Pi}^1$	
0	0	1	3.5	2.5	9.41



$E^1 = 0$
9.41
$r^1=0$
$r^1=1$
$\hat{r}^1=2$
$r^1=3$
8.1
8.82
9.41
6.05
$r^1=0$
1
$r^1=0$
1/2
$r^1=1$
1/2
$\hat{r}^1=1$
1/2
$\hat{r}^1=2$
1/2
$r^1=2$
1/2
$r^1=3$
3/8
$r^1=3$
1/8
$E^2 = 0$
$\hat{E}^2 = 1$
$\hat{E}^2 = 2$
$E^2 = 3$
$E^2 = *$
$\alpha v^2(E)$
8.1
7.54
4.28
3.75
0
$r^2 = 3$
$\hat{r}^2 = 3$
$\hat{r}^2 = 3$
$r^2 = 3$
Period 1
Period 2

Figure 4.4

Table 4.8. Probability Distribution of State of Cumulative Extraction Under the Optimal Strategy

End of Period 1

	State	Probabilities
Non-depleted States of Cumulative Extraction	0	0
	1	1/2
	2	1/2
	3	0
	4	0
	5	0
Exhaustion State	*	0

End of Period 2

	State	Probabilities
Non-depleted States of Cumulative Extraction	0	0
	1	0
	2	0
	3	2/16
	4	3/16
	5	1/16
Exhaustion State	*	10/16

tainty be resolved to determine the level of operations of its productive activities as well as its demand for the alternative primary supplies (4.16). When this is not the case, the possibilities for a specification of the supplier's revenue functions by an economic sector protecting itself against supply fluctuations are manifold. We shall briefly explore here two alternatives resulting from 1) arbitrarily penalizing the supplier for deviations from the announced supply or target extraction, and 2) an active two-stage approach similar to the case of uncertain end-use demands in section 4.III.

In the first case the supplier's revenue function would depend on both actual and target extraction denoted, as before, κ and r respectively, of the form

$$\beta^t(\kappa,r) = [\Phi^t(0) - \Phi^t(r)] + \bar{\delta}^t[\kappa - r]_+ + \underline{\delta}^t[\kappa - r]_- \qquad (4.17)$$

where $\bar{\delta}^t$ and $\underline{\delta}^t$ are non-negative per unit penalties set so as to give

$$\beta^t(\kappa,r) \le \Phi^t(0) - \Phi^t(\kappa).$$

If the actual extraction coincides with the target extraction, $\kappa = r$, the supplier will receive exactly the economic sector's cost-savings. The economic sector's cost-savings function would act as an envelope for the revenue functions (4.17) as depicted in Figure 4.5.

The sectoral problems generating the cost function Φ^t can also provide useful information for setting those penalties. If these are set such that

$$\bar{\delta}^t = -\frac{d\Phi^t}{dr}\Big|_{r=\bar{\kappa}} \quad \text{and} \quad \underline{\delta}^t = -\frac{d\Phi^t}{dr}\Big|_{r=\underline{\kappa}}$$

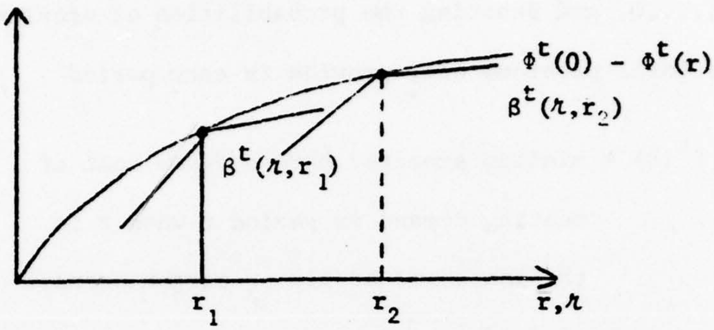


Figure 4.5

or, in other words, for general convex functions, Φ^t, as any subgradients of $\Phi^t(\bar{\textit{ʀ}})$ and $\Phi^t(\underline{\textit{ʀ}})$ respectively, the difference between what the supplier would receive had he announced $\textit{ʀ}(\Phi^t(0) - \Phi^t(\textit{ʀ}))$ and what is actually received $(\beta^t(\textit{ʀ},r))$ are maximized respectively at $\textit{ʀ} = \bar{\textit{ʀ}}$ and $\textit{ʀ} = \underline{\textit{ʀ}}$. However the solution of the supplier's problem with revenue functions given by (4.17) would be more complex because we need introduce binary (0,1) integer variables as indicators for the penalties $\bar{\delta}^t$ and $\underline{\delta}^t$.

The second possibility to be explored here for a supplier's active behavior is the derivation of a *here-and-now* expected cost function but with respect to the depletable resource supply in contrast with section 4.III. For the sake of simplicity we assume that the economic sector cannot wait and see the occurrence of the actual extraction because some of the alternative primary supplies need be ordered well in advance. For convenience we will illustrate the two stage approach in a linear programming context. Assuming that the actual supply for a fixed value of the target extraction can take only a finite number of values in the form of

$$\textit{ʀ} = \theta_q r$$

for $q=1,2,\ldots,Q$, and denoting the probabilities of occurrence p_q, the *here-and-now* sectoral problems will provide in each period

$\Gamma^t(r)$ = minimum expected *here-and-now* cost of meeting demand in period t when r is the announced supply or target extraction in period t.

Partition in (2.10) the vector of alternative primary supplies $s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ where s_1 denotes the primary supplies that need be determined in the first stage and s_2 those that can be freely set ex-post. Correspondingly, let

$$A_1^t = \begin{bmatrix} A_{11}^t \\ \\ A_{12}^t \end{bmatrix}, \; f^t = \begin{bmatrix} f_1^t, f_2^t \end{bmatrix} \text{ and } \mathcal{s}^t = \begin{bmatrix} \mathcal{s}_1^t \\ \\ \mathcal{s}_2^t \end{bmatrix}.$$

We formulate then the *here-and-now* sectoral problem as

$$\Gamma^t(r) = \min f_1^t s_1^t + \sum_{q=1}^{Q^t} p_q^t[c^t x_q^t + f_2^t s_{2q}^t]$$

$$\begin{array}{llllll}
\text{s.t.} & s_1^t \le \mathcal{s}_1^t & & & & \text{(first stage)} \\
& & \rho^t x_q^t & & \le \theta_q r & q=1,2,\ldots,Q^t \\
& & & & & (4.18) \\
& -s_1^t + & A_{11}^t x_q^t & & \le 0 & q=1,2,\ldots,Q^t \\
& & A_{12}^t x_q^t & -s_{2q}^t & \le 0 & q=1,2,\ldots,Q^t \\
& & & & & \text{(second stage)} \\
& & A_2^t x_q^t & & \ge d^t & q=1,2,\ldots,Q^t \\
& & x_q^t \ge 0 & 0 \le s_{2q}^t \le \mathcal{s}_2^t & & q=1,2,\ldots,Q^t
\end{array}$$

Because of the linearity in r in (4.18), Γ^t will be non-increasing, convex, and piecewise linear with a finite number of segments.

For comparative purposes, we let for $q=1,2,\ldots,Q^t$

$$\begin{aligned}
\Phi_q^t(r) = \min\ & c^t x_q^t + f^t s_q^t \\
\text{s.t. } & \rho^t x_q^t \le \theta_q r \\
& A_1^t x^t - s_q^t \le 0 \\
& A_2^t x_q^t \ge d^t \\
0 \le x_q^t \quad & 0 \le s_q^t \le \bar{s}^t
\end{aligned}$$

denote the cost of meeting demand when r is the announced target supply, for every possible occurrence of θ. If the economic sector could *wait and see*, the realization of the actual supplier would be given by

$$\sum_{q=1}^{Q^t} p_q^t[\Phi_q^t(0) - \Phi_q^t(r)].$$

We can then show

<u>Lemma 4.3</u>: $\Gamma^t(0) = \sum_{q=1}^{Q^t} p_q^t \Phi_q^t(0)$ and

$$\Gamma^t(r) \ge \sum_{q=1}^{Q^t} p_q^t \Phi_q^t(r) \quad \text{for } r \ge 0.$$

Proof: The proof of this lemma can be carried in an analogous manner of Lemma 4.2 and shall not be repeated here.||

If *here-and-now* the maximum amount the sector is willing to pay the supplier for r units of the depletable resource is $\Gamma^t(0) - \Gamma^t(r)$, and

$$\beta^t(r) \equiv \Gamma^t(0) - \Gamma^t(r),$$

the supplier is already being charged the sector's cost of uncertainty,

$$\Gamma^t(r) - \sum_{q=1}^{Q^t} p_q^t[\Phi_q^t(0) - \Phi_q^t(r)],$$

since by Lemma 4.3

$$\beta^t(r) \equiv \Gamma^t(0) - \Gamma^t(r) \leq \sum_{q=1}^{Q^t} p_q^t[\Phi^t(0) - \Phi_q^t(r)].$$

Chapter 5: Models of Depletable Resources over an Infinite Horizon

5.I. Introduction

In our development so far the planning horizon for the depletable resource supplier(s) was assumed finite. No criterion or assumption has been placed in the determination of the length of this planning horizon. It is widely recognized (see section 1.II.2) that unless minimum resource requirements exist in the economy, the relevant planning horizon for the depletable resource is for all purposes, infinite. However, our finite-horizon analysis had enough flexibility to accommodate the infinite-horizon problem by a proper selection of the salvage valuation. This was the topic of our discussion at the end of Chapter 2. In this chapter we consider some infinite-horizon depletable resource planning models. In the next two sections we focus on stationary and semi-stationary infinite-horizon extensions of equilibrium in our basic model. This is followed by a model of sectoral ownership of the depletable resource reserves under an infinite planning horizon and Leontief substitution systems. A digression at the end of the chapter suggests an ascent algorithm for non-stationary infinite horizon problems.

5.II. Finite-Horizon Approximations to the Stationary Infinite-Horizon Supplier's Problem

In section 2.VI we mentioned the possibility of extending the sectoral-supplier depletable resource equilibrium to an infinitely long planning horizon. Under non-stationary conditions this would require computing the sectoral cost of meeting demand, Φ^t, by solving an infinite number of

sectoral problems which is obviously undesirable. Even under conditions of perfect stationarity, solving the infinite-horizon supplier's problem is, in most cases, non-trivial. In the present section we discuss two finite-horizon approximation methods to the *stationary infinite-horizon supplier's problem*

$$V(S) = \max \sum_{t=1}^{\infty} \alpha^{t-1} \{\beta(r^t) - g\,(\sum_{j=1}^{t-1} r^j, r^t)\}$$

$$\text{s.t.} \quad \sum_{t=1}^{\infty} r^t \leq S \tag{5.1}$$

$$r^t \geq 0 \qquad t=1,2,\ldots\ldots\ldots\ldots$$

where S denotes the initial state.

In one case the infinite horizon is truncated at the end of a finite period. The successive *finite-horizon truncations* will provide monotonically increasing lower bounds to the optimal value of the original problem and a sequence of feasible solutions to the stationary infinite-horizon problem. The second finite-horizon approximation method consists essentially of a relaxation of the original problem. The sequence of *finite-horizon α-relaxations*, on the other hand, will provide monotonically decreasing upper bounds to the optimal value of the stationary infinite-horizon supplier's problem (5.1), but the sequence of solutions will not be feasible to the stationary infinite-horizon problem necessarily.

The stationary infinite-horizon supplier's problem (5.1) can be reformulated in terms of the dynamic programming backward recursions

$$V(S) = \sup_{0 \leq r \leq S} \{\beta(r) - g(R-S,r) + \alpha\, V(S-R)\}. \qquad (5.2)$$

The solution to the functional equation (5.2), evaluated at the point $S=\underset{\sim}{S}$. is a solution of (5.1). We can then show

Lemma 5.1: There exists a unique function V satisfying the functional equation (5.2)

Proof: By verifying that the payoff function

$$\beta(r) - g(R-S,r) + \alpha\, V(S-r)$$

satisfies Denardo's [16] monotonicity and contraction assumption, we can rely on the elegant and more general proof of his Theorem 3, to ascertain the existence of a unique solution to (5.2)

Lemma 5.2: V(S) given by (5.2) is

(i) concave if β is concave and g is convex

(ii) non-decreasing if g is non-decreasing in its first argument

(iii) continuous if β and g are continuous.

Proof: Consider the method of successive approximations with V_n denoting the n-th approximation and

$$V_n(S) = \sup_{0 \leq r \leq S} \{\beta(r) - g(R-S,r) + \alpha\, V_{n-1}(S-r)\} \qquad n=1,2,\ldots$$

Denardo [16] has shown that

$$\lim_{n\to\infty} V_n = V$$

and that convergence is uniform for any initial V_o. It was shown in Lemma 2.3 that for concave β and V_{n-1} and convex g, V_n is concave. Since for any $0\leq\omega\leq 1$

$$V_n[\omega S_1+(1-\omega)\ S_2] \geq \omega V_n(S_1) + (1-\omega)\ V_n(S_2),$$

taking limits on both sides yields

$$V[\omega S_1 + (1-\omega)\ S_2] \geq \omega V(S_1) + (1-\omega)\ V(S_2)$$

which completes the proof of (i).

It is trivial to show that if V_{n-1} is non-decreasing and g is non-decreasing in its first argument then for $S_1 \geq S_2$

$$V_n(S_1) \geq V_n(S_2)\ .$$

Taking limits on both sides gives

$$V(S_1) \geq V(S_2)$$

which yields (ii).

If β,g and V_{n-1} are continuous then the payoff function

$$\beta(r) - g(R-S,r) + \alpha\ V_{n-1}(S-r) \qquad (5.3)$$

is continuous on S and r. Since S ε[0,R], a compact set, and the point-

to-set mapping, $\Omega(S) = [0,S]$ is continuous, then $V_n(S)$, the maximum of (5.3) over $r\varepsilon\Omega$ (S) is from Meyer's ([64], theorem 1.4) continuous. Since convergence is uniform, V(S) is continuous. ||

The first approximation method we suggest is the finite horizon truncation of the infinite-horizon, in which case, we solve the *T-period truncation*

$$\underset{\sim}{V}_T(S) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta(r^t) - g(\sum_{j=1}^{t-1} r^j, r^t)\}$$

$$\text{s.t.} \quad \sum_{t=1}^{T} r^t \leq S \tag{5.4}$$

$$r^t \geq 0 \qquad t=1,2,\ldots,T$$

which is essentially the finite-horizon supplier's problem (2.1) with no salvage function ($\beta^{T+1}(S) = 0$, for all S). For problem (5.4) we can show

<u>Lemma 5.3</u>: (i) $\underset{\sim}{V}_T(S) \leq \underset{\sim}{V}_{T+1}(S) \leq V(S)$

(ii) $$\underset{\sim}{V}_T^t(S) = \max_{0\leq r\leq S} \{\beta(r) - g(R-S,r) + \alpha \underset{\sim}{V}_T^{t+1}(S-r)\}$$

$$t=T,T-1,\ldots,1$$

with

$$\underset{\sim}{V}_T^{T+1}(S) = 0 \qquad \text{for all S}$$

and, $\underset{\sim}{V}_T(S) \equiv \underset{\sim}{V}_T^1(S)$

$$\text{(iii)}\quad \underset{\sim}{V}_{T+1}(S) = \max_{0 \leq r \leq S} \{\beta(r) - g(R-S,r) + \alpha\underset{\sim}{V}_T(S-r)\}$$

$$T=1,2,\ldots$$

Proof: Let $\hat{r}_T^t$, t=1,2,...,T, be an optimal solution to the T-period truncation (5.4) with value $\underset{\sim}{V}_T(S)$. Then

$$r_{T+1}^t = \begin{cases} \hat{r}_T^t & t=1,2,\ldots,T \\ 0 & t = T+1 \end{cases}$$

is a feasible solution to the T+1-period truncation with value $\underset{\sim}{V}_T(S)$. We then have

$$\underset{\sim}{V}_T(S) \leq \underset{\sim}{V}_{T+1}(S) \quad .$$

For any T=1,2,...,

$$r^t = \begin{cases} \hat{r}_T^t & t=1,2,\ldots,T \\ 0 & t=T+1,T+2,\ldots \end{cases}$$

is feasible for the infinite-horizon supplier's problem (5.1) with value $\underset{\sim}{V}_T(S)$. Consequently

$$\underset{\sim}{V}_T(S) \le \underset{\sim}{V}(S)$$

for all T. This completes the proof of (i). Part (ii) is just a dynamic programming reformulation of (5.4), in terms of a backward recursion. Part (iii) gives the T+1 period truncation recursively as a function of the T-period truncation. ||

Concerning the convergence of the T-period truncations (5.4) we can show

<u>Theorem 5.1</u>: $\lim_{T\to\infty} \underset{\sim}{V}_T(S) = V(S)$

Proof: From Lemma 5.3(i), the sequence $\underset{\sim}{V}_T(\ )$ converges and we have

$$W(S) = \lim_{T\to\infty} \underset{\sim}{V}_T(S) \le V(S) \quad .$$

We need to show that this relation holds as an equality. Let

$$\underset{\sim}{\varepsilon}_{T+1}(S) = \underset{\sim}{V}_{T+1}(S) - V(S)$$

denote the error of the T+1-period truncation. Then by Lemma 5.3 (iii)

$$\underset{\sim}{\varepsilon}_{T+1}(S) = \max_{0\le r\le S} \{\beta(r) - g(R-S,r) + \alpha\underset{\sim}{V}_T(S-r)\}$$

$$- \max_{0\le r\le S} \{\beta(r) - g(R-S,r) + V(S-r)\}$$

Let $\hat{r}$ be an optimal solution to the second expression, then

$$\underset{\sim}{\varepsilon}_{T+1}(S) = \max_{0\le r\le S} \{\beta(r) - \beta(\hat{r}) - g(R-S,r) + g(R-S,\hat{r})$$

$$+ \alpha[\underset{\sim}{V}_T(S-r) - V(S-\hat{r})]\} \qquad (5.5)$$

$$= \max_{0\le r\le S} \{\alpha\underset{\sim}{\varepsilon}_T(S-r) + \beta(r) - g(R-S,r) + \alpha V(S-r)$$

$$- \beta(\hat{r}) - g(R-S,\hat{r}) + \alpha V(S-\hat{r})\}$$

Since $0\le r\le S$

$$\beta(\hat{r}) - g(R-S,\hat{r}) + \alpha V(S-\hat{r}) \ge \beta(r) - g(R-S,r) + \alpha V(S-r)$$

we have[1]

$$\underset{\sim}{\varepsilon}_{T+1}(S) \le \alpha \max_{0\le r\le S} \underset{\sim}{\varepsilon}_T(S-r) \le \alpha \,||\underset{\sim}{\varepsilon}_T|| \quad . \qquad (5.6)$$

Also since $r = \hat{r}$ is feasible for (5.5)

$$\underset{\sim}{\varepsilon}_{T+1}(S) \ge \alpha\underset{\sim}{\varepsilon}_T(S-\hat{r}) \ge -\alpha||\underset{\sim}{\varepsilon}_T|| \quad . \qquad (5.7)$$

From (5.6) and (5.7)

$$||\underset{\sim}{\varepsilon}_{T+1}|| \le \alpha||\underset{\sim}{\varepsilon}_T||$$

[1] $||\varepsilon||$ denotes the sup norm.

Consequently

$$||\underset{\sim}{\varepsilon}_{T+1}|| \leq \alpha^T \, ||\underset{\sim}{\varepsilon}_1|| \quad , \qquad T=1,2,\ldots,$$

and since $\alpha < 1$

$$\lim_{T\to\infty} ||\underset{\sim}{\varepsilon}_{T+1}|| = 0$$

we conclude then that

$$W(S) = \lim_{T\to\infty} \underset{\sim}{V}_T(S) = V(S) \quad . \, ||$$

<u>Lemma 5.4</u>: If in a solution of the infinite-horizon supplier's problem $\hat{r}^t = 0$, for $t \geq T+1$, the solution to the finite-horizon truncations (5.4) converges to an optimal solution of the infinite-horizon supplier's problem (5.1) after at most T truncations.

Proof: This follows by observing that

$$r_T^t = \hat{r}^t \qquad\qquad t=1,2,\ldots,T$$

is feasible for the T-period truncation with value $V(S)$. Hence

$$V(S) \leq \underset{\sim}{V}_T(S) \quad .$$

By Lemma 5.3, we must have

$$V(S) = \underset{\sim}{V}_T(S) \quad .$$

Consequently $(r_T^1, r_T^2, \ldots, r_T^T)$ is an optimal solution to (5.4). ||

While the finite-horizon truncation of the infinite planning horizon described above seems to be generally applicable, the next approximation method to be discussed, due to Grinold [31] was designed primarily for a class of infinite-horizon stationary linear programs. For piecewise linear revenue function β and a "cumulative" extraction cost function e, we shall show that the stationary infinite-horizon supplier's problem (5.1) can be formulated such as to fit that category. Before introducing the finite-horizon α-relaxations we need introduce the concept of α-*convergence*. We will say that the sequence $\{r^t\}$ is α-convergent if each vector of the sequence is non-negative and the infinite sum $\sum_{t=1}^{\infty} \alpha^{t-1} r^t$ converges. We can then show

<u>Lemma 5.5</u>: If the sequence $\{r^t\}$ is bounded then it is α-convergent.

Proof: Let $0 \leq r^t \leq S$ for t-1,2,..., then

$$\sum_{t=1}^{T} \alpha^{t-1} r^t \leq \frac{S}{1-\alpha} \cdot ||$$

What the above lemma shows is that the reserves constraint in the supplier's problem induces α-convergence on the set of feasible solutions. In order to relate the stationary infinite-horizon supplier's problem (5.1) with the class of problems studied by Grinold [31] we rewrite it in terms of S^t, the resource stock left by the start of period t.

$$V(S) = \max \sum_{t=1}^{\infty} \alpha^{t-1} \{\beta(r^t) - (1-\alpha)\, e(R-S^t+r^t)\}$$

$$\text{s.t.} \quad S^1 = S \qquad (5.8)$$

$$S^t = S^{t-1} - r^{t-1} \qquad t=2,3,\ldots,$$

$$S^t \geq 0,\ r^t \geq 0 \qquad t=2,3,\ldots,$$

Similarly to the finite-horizon development in section 2.IV.2, when

$$\beta(r) = \min_{k=1,2,\ldots,K} \{b^k + \pi^k r\}$$

$$e(E) = \max_{j=1,2,\ldots,J} \{e^j + \gamma^j E\}$$

problem (5.8) can be rewritten as an infinite-horizon linear program with

$$V(S) = \max \sum_{t=1}^{\infty} \alpha^{t-1} \{z^t - (1-\alpha)\, y^t\}$$

$$
\begin{array}{lll}
\text{s.t} \quad z^1 - \pi^k r^1 \le b^k & & k=1,2,\ldots K \\
y^1 - \gamma^j r^1 + \gamma^j S^1 \ge e^j + \gamma^j{}_r & & j=1,2,\ldots J \\
S^1 = S & & (5.9) \\
z^t - \pi^k r^t \le b^k & & k=1,2,\ldots,K \\
y^t - \gamma^j r^t + \gamma^j S^t \ge e^j + \gamma^j R & & j=1,2,\ldots,J \\
 & & t=2,3,\ldots, \\
S^t = S^{t-1} - r^{t-1} & & t=2,3,\ldots, \\
S^t \ge 0 \ , \ r^t \ge 0 & &
\end{array}
$$

By introducing slack and surplus variables in (5.9) and letting

$$
A = \begin{bmatrix}
1 & 0 & -\pi^1 & 0 & 1 & 0\ldots\ldots 0 & 0 & 0\ldots\ldots 0 \\
1 & 0 & -\pi^2 & 0 & 0 & 1\ldots\ldots 0 & 0 & 0\ldots\ldots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots \\
1 & 0 & -\pi^K & 0 & 0 & 0\ldots\ldots 1 & 0 & 0\ldots\ldots 0 \\
0 & 1 & -\gamma^1 & \gamma^1 & 0 & 0\ldots\ldots 0 & -1 & 0\ldots\ldots 0 \\
0 & 1 & -\gamma^2 & \gamma^2 & 0 & 0\ldots\ldots 0 & 0 & -1\ldots\ldots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots \\
0 & 1 & -\gamma^J & \gamma^J & 0 & 0\ldots\ldots 0 & 0 & 0\ldots\ldots -1 \\
0 & 0 & 0 & 1 & 0 & 0\ldots\ldots 0 & 0 & 0\ldots\ldots 0
\end{bmatrix}
$$

$$
p = \begin{bmatrix}
1 & -(1-\alpha) & 0 & 0 & 0 & 0\ldots\ldots 0 & 0 & 0\ldots\ldots 0
\end{bmatrix}
$$

$$K=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0\\
0 & 0 & 0 & 0 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\quad\;\;\vdots & \vdots & \vdots\quad\;\;\vdots\\
0 & 0 & 0 & 0 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0\\
0 & 0 & 0 & 0 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0\\
0 & 0 & 0 & 0 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\quad\;\;\vdots & \vdots & \vdots\quad\;\;\vdots\\
0 & 0 & 0 & 0 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0\\
0 & 0 & -1 & 1 & 0 & 0\cdots\cdots 0 & 0 & 0\cdots\cdots 0
\end{bmatrix}$$

$$x^t=\begin{bmatrix} z^t\\ y^t\\ r^t\\ S^t\\ \zeta^1\\ \vdots\\ \zeta^K\\ \zeta^1\\ \vdots\\ \zeta^J \end{bmatrix}\qquad
b=\begin{bmatrix} b^1\\ b^2\\ \vdots\\ b^K\\ e^1+\gamma^1R\\ e^2+\gamma^2R\\ \vdots\\ e^J+\gamma^JR\\ 0 \end{bmatrix}\qquad
s=\begin{bmatrix} b^1\\ b^2\\ \vdots\\ b^K\\ e^1+\gamma^1R\\ e^2+\gamma^2R\\ \vdots\\ e^J+\gamma^JR\\ S \end{bmatrix}$$

problem (5.9) is of the form

$$V(s) = \max \sum_{t=1}^{\infty} \alpha^{t-1} p\, x^t$$

s.t.

$$Ax^1 = s \tag{5.10}$$

$$Ax^t = b+K\, x^{t-1} \qquad t=2,\ldots\ldots$$

$$x^t \geq 0$$

Grinold [31] suggested the following T-period approximations to solve (5.10)

$$\tilde{V}_T(s) = \max \sum_{t=1}^{T} \alpha^{t-1} p\, x_T^t$$

$$\text{s.t.} \qquad Ax_T^1 = s$$

$$Ax_T^t = b + K\, x_T^{t-1} \qquad t=2,3,\ldots,T-1 \tag{5.11}$$

$$(A-\alpha K)x_T^T = \frac{b}{1-\alpha} + K\, x_T^{t-1}$$

$$x_T^t \geq 0 \qquad t=1,2,\ldots,T$$

In terms of our original problem (5.8) Grinold's [31] T-period approximations (5.11) would result in the *T-period* α*-relaxation*

$$\tilde{V}_T(S) = \max \sum_{t=1}^{T} \alpha^{t-1} \{\beta_T^t(r_T^t) - (1-\alpha)\, e_T^t(R-S_T^t+r_T^t)\}$$

$$S_T^1 = S \tag{5.12}$$

$$S_T^t - S_T^{t-1} - r_T^{t-1} \qquad t=2,3,\ldots,T-1$$

$$(1-\alpha)S_T^T + \alpha r_T^T = S_T^{T-1} - r_T^{T-1}$$

$$0 \leq S_T^t \ , \ 0 \leq r_T^t \qquad t=1,2,\ldots,T$$

where

$$\beta_T^t(r) = \min_{k=1,2,\ldots,K} \{b^k + \pi^k r\} \qquad t=1,2,\ldots,T-1$$

$$\beta_T^T(r) = \min_{k=1,2,\ldots,K} \{\frac{b^k}{1-\alpha} + \pi^k r\} \tag{5.13}$$

$$e_T^t(R-S_T^r+r_T^t) = \max_{j=1,2,\ldots,J} \{e^j+\gamma^j \ [R-S_T^t+r_T^t]\} \qquad t=1,2,\ldots,T-1$$

$$e_T^T(R-S_T^T+r_T^T) = \max_{j=1,2,\ldots,J} \{\frac{e^j+\gamma^j R}{1-\alpha} - \gamma^j S_T^T + \gamma^j r_T^T\} \tag{5.14}$$

requiring only a slight modification of the revenue function (5.13) and extraction cost function (5.14) for the last period in the approximation.

<u>Lemma 5.6</u>: (i) $\tilde{V}_T(S) \geq \tilde{V}_{T+1}(S) \geq V(S)$

(ii) $\tilde{V}_T^t(S) = \max\limits_{0 \leq r_T \leq S} \{\beta_T^t(r_T) - (1-\alpha)\, e_T^t(R-S+r_T) +$

$$+ \alpha\tilde{V}_T^{t+1}(S-r_T)\}$$

$$t=T-1,T-2,\ldots,1$$

with $$\tilde{V}_T^T(S) = \max\{\beta_T^T(r_T) - (1-\alpha) \quad e_T^T(R-S_T^T+r_T)\}$$

$$\text{s.t.} \qquad \alpha r_T + (1-\alpha)\, S_T^T = S$$

and

$$\tilde{V}_T(S) \equiv \tilde{V}_T^1(S)$$

(iii) $$\tilde{V}_{T+1}(S) = \max_{0\le r\le S} \{\beta(r) - (1-\alpha) \quad e(R-S+r) + \alpha\tilde{V}_T(S-r)\}$$

$$T=1,2,3,\ldots,$$

Proof: Let the pair $(\hat{r}_{T+1}^t, \hat{S}_{T+1}^t)$, $t=1,2,\ldots,T+1$, be an optimal solution to the T+1-*period* α-*relaxation* (5.12). For

$$r_T^t = \begin{cases} \hat{r}_{T+1}^t & t=1,2,\ldots,T-1 \\ \hat{r}_{T+1}^T + \alpha\hat{r}_{T+1}^{T+1} & t=T \end{cases}$$

$$S_T^t = \begin{cases} \hat{S}_{T+1}^t & t=1,2,\ldots,T-1 \\ \hat{S}_{T+1}^T + \alpha \hat{S}_{T+1}^{T+1} & t=T \end{cases}$$

we have $\qquad S_T^1 = S$

$$S_T^t = \hat{S}_{T+1}^t = \hat{S}_{T+1}^{t-1} - \hat{r}_{T+1}^{t-1} = S_T^{t-1} - r_T^{t-1} \qquad t=2,3,\ldots,T-1$$

$$\alpha r_T^T + (1-\alpha)\; S_T^T = \alpha \quad \alpha \hat{r}_{T+1}^{T+1} + (1-\alpha)\; \hat{S}_{T+1}^{T+1} \quad + \alpha \hat{r}_{T+1}^T + (1-\alpha)\; \hat{S}_{T+1}^T$$

$$= \hat{S}_{T+1}^T = \hat{S}_{T+1}^{T-1} - \hat{r}_{T+1}^{T-1} = S_T^{T-1} - r_T^{T-1}$$

and then (r_T^t, S_T^t) is feasible for the T-period α-relaxation (5.12). For the value of this feasible solution we show

$$\beta_T^t(r_T^t) = \beta_T^t(\hat{r}_{T+1}^t) = \beta_{T+1}^t(\hat{r}_{T+1}^t) \qquad t=1,2,\ldots,T-1$$

$$\beta_T^T(r_T^T) = \min_{k=1,\ldots,K} \left\{ \frac{b^k}{1-\alpha} + \pi^k\, r_T^T \right\} =$$

$$\min_{k=1,\ldots,K} \left\{ b^k + \pi^k\, \hat{r}_{T+1}^T + \alpha\; \frac{b^k}{1-\alpha} + \pi^k\, \hat{r}_{T+1}^{T+1} \right\} \geq$$

$$\min_{k=1,\dots,K}\left\{b^k + \pi^k\hat{r}^T_{T+1}\right\} + \alpha \min_{k=1,\dots,K}\left\{\frac{b^k}{1-\alpha} + \pi^k\hat{r}^{T+1}_{T+1}\right\} = \beta^T_{T+1}(\hat{r}^T_{T+1}) + \alpha\beta^{T+1}_{T+1}(\hat{r}^{T+1}_{T+1})$$

$$e^t_T(R-S^t_T+r^t_T) = e^t_T(R-\hat{S}^t_{T+1}+\hat{r}^t_{T+1}) = e^t_{T+1}(R-\hat{S}^t_{T+1}+\hat{r}^t_{T+1}) \qquad t=1,2,\dots,T-1$$

$$e^T_T(R-S^T_T+r^T_T) = \max_{j=1,\dots,J}\left\{\frac{e^j+\gamma^j R}{1-\alpha} - \gamma^j S^T_T + \gamma^j r^T_T\right\} =$$

$$\max_{j=1,\dots,J}\left\{e^j + \gamma^j \quad R-\hat{S}^T_{T+1}+\hat{r}^T_{T+1} \quad + \alpha \quad \frac{e^j+\gamma^j R}{1-\alpha} - \gamma^j\hat{S}^{T+1}_{T+1} + \gamma^j\hat{r}^{T+1}_{T+1}\right\} \le$$

$$\max_{j=1,\dots,J}\left\{e^j + \gamma^j \quad R-\hat{S}^T_{T+1}+\hat{r}^T_{T+1}\right\} + \alpha \max_{j=1,\dots,J}\left\{\frac{e^j+\gamma^j R}{1-\alpha} - \gamma^j\hat{S}^{T+1}_{T+1} + \gamma^j\hat{r}^{T+1}_{T+1}\right\} = e^T_{T+1}(R-\hat{S}^T_{T+1}+\hat{r}^T_{T+1}) + \alpha e^{T+1}_{T+1}(R-\hat{S}^{T+1}_{T+1}+\hat{r}^{T+1}_{T+1}) \ .$$

Hence the value of this feasible solution is greater than or equal to $\tilde{V}_{T+1}(S)$. Consequently,

$$\tilde{V}_T(S) \ge \tilde{V}_{T+1}(S) \ .$$

Let (r^t,s^t), $t=1,2,\dots,$ be an optimal solution to the stationary infinite-horizon supplier's problem (5.8). We may then show that the pair

$$r_T^t = \begin{cases} \hat{r}^t & t=1,2,\ldots,T-1 \\ \sum_{t=T}^{\infty} \alpha^{t-T}\hat{r}^t & t=T \end{cases}$$

$$S_T^t = \begin{cases} \hat{S}^t & t=1,2,\ldots,T-1 \\ \sum_{t=T}^{\infty} \alpha^{t-T}\hat{S}^t & t=T \end{cases}$$

is feasible for the T-period α-relaxation (5.12). As above, the value of this feasible solution can be shown to be greater or equal to $V(\mathcal{S})$. These series converge as a result of Lemma 5.6. Consequently,

$$\tilde{V}_T(\mathcal{S}) \geq V(\mathcal{S})$$

which completes the proof of (i).

In (ii) the finite-horizon α-relaxations (5.12) are rewritten as a dynamic programming problem through a backward recursive functional equation. Part (iii) gives the T+1 period α-relaxation recursively as a function of the T-period α-relaxation. ||

<u>Theorem 5.2</u>: $\lim_{T\to\infty} V_T(\mathcal{S}) = V(\mathcal{S})$.

Proof: See Grinold [31]. ||

<u>Lemma 5.7</u>: If in a solution to the T-period α-relaxation (5.12), $r_T^T = 0$, then the pair

$$r^t = \begin{cases} \hat{r}_T^t & t=1,2,\ldots,T-1 \\ 0 & t \geq T \end{cases}$$

$$S^t = \begin{cases} \hat{S}_T^t & t=1,2,\ldots,T-1 \\ \hat{S}_T^T & t \geq T \end{cases}$$

is an optimal solution to the stationary infinite-horizon supplier's problem (5.8).

Proof: This follows by showing that the pair (r^t, S^t) is feasible for (5.8), with

$$\beta(r^t) = \beta(\hat{r}_T^t) = \beta_T^t(\hat{r}_T^t) \qquad t=1,2,\ldots,T-1$$

$$\beta(r^t) = 0 \qquad t \geq T$$

$$e(R-S^t+r^t) = e(R-\hat{S}_T^t+\hat{r}_T^t) = e_T^t(R-\hat{S}_T^t+\hat{r}_T^t) \qquad t=1,2,\ldots,T-1$$

$$e(R-S^t+r^t) = e(R-\hat{S}_T^T) \quad . \qquad t \geq T$$

Hence, the value of this feasible solution is $\tilde{V}_{T+1}(S)$ and therefore

$$V(S) \geq \tilde{V}_{T+1}(S)$$

By Lemma 5.6, it must be that

$$V(S) = \tilde{V}_{T+1}(S)$$

which completes the proof. ||

What the above corollary gives is a finite stopping criterion that guarantees that an optimal solution to the infinite horizon problem has been reached. Notice that the converse statement, namely that if in the optimal solution to the infinite horizon problem $r^t = 0$ for $t \geq T$ this solution would be reached after at most T+1 approximations has not been shown. It is conceivable that this cannot be shown without resorting to considerations over the dual problem to (5.11). This would make the notation unnecessarily complicated.

In the solution to the stationary infinite-horizon supplier's problem, the combination of the finite-horizon truncations (5.4) with the finite-horizon α-relaxation (5.12), is insightful in terms of providing bounds on the maximum error of either the T-period truncation or of the T-period α-relaxation. The combination of Lemmas 5.3 and 5.6 yields

$$\tilde{V}_T(S) \geq \tilde{V}_{T+1}(S) \geq V(S) \geq \underset{\sim}{V}_{T+1}(S) \geq \underset{\sim}{V}_T(S)$$

and therefore

$$\tilde{\varepsilon}_{T+1}(S) = \tilde{V}_{T+1}(S) - V(S) \leq \tilde{V}_{T+1}(S) - \underset{\sim}{V}_{T'}(S) \qquad T'=1,2\ldots$$

$$\underset{\sim}{\varepsilon}_{T+1}(S) = V(S) - \underset{\sim}{V}_T(S) \leq \tilde{V}_{T'+1}(S) - \underset{\sim}{V}_T(S) \qquad T'=1,2,\ldots$$

Example 5.1: As an illustrating example of the application of finite-horizon approximations, we assume the conditions of the stationary Example 1.2. The revenue function β is given in Tab. 2.5. Extraction costs are given by $g(E,r) = r$ for all $0 \leq E \leq R$ and are therefore independent of past cumulative extraction which makes this example a further simplified version of problem (5.1) or (5.8). The discount rate is $\alpha = .9$ and $R = 213$.

The sequence of finite-horizon truncations (5.4) is depicted in Table 5.1. A comparison of Tbs. 5.1 and 2.7,shows convergence after the eight-period truncation (T=8). This was expected as a result of Lemma 5.4. For the finite-horizon α-relaxations (5.12), the modified last-period function β_T^T, as defined in (5.13) is depicted in Table 5.2. From Tab. 5.3 and Lemma 5.7, we would stop after the nine-period approximation (T=9). A comparison of Tbs. 5.3 and 2.7confirms that an optimal solution to the stationary infinite-horizon supplier's problem has been reached.

Tab. 5.4 shows the errors in the sequence of finite-horizon approximations. Notice that in this example the errors from the finite-horizon truncations are in absolute value consistently larger than the errors from the finite-horizon α-relaxation except at T=8. The relaxations need one more approximation to converge. For T=5, the finite-horizon α-relaxations are already within 7% of the value of the optimal infinite-horizon solution.

Table 5.1: Finite-Horizon Truncations

Length of Truncation T	1	2	3	4	5	6	7	8	9	10
$\underset{\sim}{V}_T$	285.95	543.30	774.92	983.38	1,170.99	1,339.84	1,491.81	1,532.15	1,532.15	1,532.15
r_T^1	30.26	30.26	30.26	30.26	30.26	30.26	30.26	28.15	28.15	28.15
r_T^2		30.26	30.26	30.26	30.26	30.26	30.26	28.15	28.15	28.15
r_T^3			30.26	30.26	30.26	30.26	30.26	28.15	28.15	28.15
r_T^4				30.26	30.26	30.26	30.26	27.41	27.41	27.41
r_T^5					30.26	30.26	30.26	27.36	27.36	27.36
r_T^6						30.26	30.26	27.36	27.36	27.36
r_T^7							30.26	27.36	27.36	27.36
r_T^8								19.06	19.06	19.06
r_T^9									0	0
r_T^{10}										0

Table 5.2: Last Period Modified Revenue Function (β_T^T)

Range of r		Shadow Prices	
Minimum	Maximum		
0	273.60	$\pi^{T,1}$	11
273.60	274.10	$\pi^{T,2}$	7.93
274.10	281.50	$\pi^{T,3}$	7.16
281.50	297.40	$\pi^{T,4}$	5.42
297.40	302.60	$\pi^{T,5}$	1.80
302.60	357.10	$\pi^{T,6}$	.40
357.10	296.90	$\pi^{T,7}$	.35
396.90	∞	$\pi^{T,8}$	0

Table 5.3: Finite-Horizon α-Relaxations

Length of Approximation T	1	2	3	4	5	6	7	8	9	10
$\tilde{V}_T$	2,366.67	2,130.99	1,944.39	1,801.93	1,695.77	1,619.94	1,570.22	1,542.14	1,532.15	1,532.15
r_T^1	236.67	27.36	27.36	27.36	27.36	27.41	28.15	28.15	28.15	28.15
r_T^2		206.27	27.36	27.36	27.36	27.36	27.41	28.15	28.15	28.15
r_T^3			175.87	27.36	27.36	27.36	27.36	27.41	28.15	28.15
r_T^4				145.47	27.36	27.36	27.36	27.36	27.41	27.41
r_T^5					115.07	27.36	27.36	27.36	27.36	27.36
r_T^6						84.61	27.36	27.36	27.36	27.36
r_T^7							53.33	27.36	27.36	27.36
r_T^8								22.06	19.06	19.06
r_T^9									0	0
r_T^{10}										0

Table 5.4: Approximation Errors

Length of Approximation T	1	2	3	4	5	6	7	8	9	10
$\underset{\sim}{V}_T$	285.95	543.30	774.92	983.38	1,170.99	1,339.84	1,491.81	1,532.15	1,532.15	1,532.15
$\tilde{V}_T$	2,366.67	2,130.09	1,944.39	1,801.93	1,695.77	1,619.94	1,570.22	1,542.14	1,532.15	1,532.15
V	1,532.15	1,532.15	1,532.15	1,532.15	1,532.15	1,532.15	1,532.15	1,532.15	1,532.15	1,532.15
$V - \underset{\sim}{V}_T$	1,246.20	988.85	757.23	548.77	361.16	192.31	40.34	0	0	0
$\tilde{V}_T - V$	834.52	597.94	412.24	269.78	106.16	87.79	38.07	9.99	0	0
$\tilde{V}_T - \underset{\sim}{V}_T$	2,080.72	1,586.79	1,169.47	818.55	524.78	280.10	78.41	9.99	0	0

5.III. Semi-Stationary Supplier's Problem

Section 5.II suggested two finite-horizon approximation methods to solve the stationary infinite-horizon supplier's problem with a given initial state, the depletable resource stock, S. When the state by the start of the stationary stage is not known a priori, generating $V(S)$ for all $0 \leq S \leq R$, by any of the above methods is not viable. We need rely on other procedures to value stocks, or in other words, to determine a solution to the functional equation

$$V(S) = \sup_{0 \leq r \leq S} \{\beta(r) - g(R-S,r) + \alpha V(S-r)\} \tag{5.15}$$

Denardo [16] suggested two techniques for approximating or determining V in (5.15): the method of successive approximations and a generalization of Howard's [45] policy improvement routine. The finite-horizon truncation of the infinite planning horizon, described in section 5.II, can be visualized in terms of these two techniques. It is a successive approximation method with an initial approximation

$$V_o(S) = 0 \quad .$$

Alternatively it corresponds to an improvement over an initial policy of setting

$$r = 0 \text{ for all } S.$$

It is conceivable that other techniques such as Scarf's [73] simplicial approximation exploring the fixed-point properties of the functional equation (5.15), may lead to efficient solution procedures.

The next approximation we describe explores the concavity and continuity of V as shown in Lemma 5.2. This consists of either the outer or inner-linearization of V. Suppose V has been determined by either the finite-horizon truncation or α-relaxation approximation methods described in section 5.II, at a set of states $0 = S^1, S^2, \ldots S^K = R$ with values V^1, V^2, $\ldots V^K$ and subgradients π^1, $\pi^2, \ldots \pi^K$. The inner-linearization of V gives

$$\underset{\sim}{V}(S) = \max \sum_k \zeta^k V^k$$

$$\text{s.t.} \sum_k \zeta^k S^k = S$$

$$\sum_k \zeta^k = 1$$

$$\zeta^k \geq 0$$

as a lower-bound approximation to V. Alternatively the outer-linearization of V gives

$$\tilde{V}(S) = \min z$$

$$\text{s.t. } z \leq V^k + \pi^k(S-S^k) \quad k-1,2,\ldots K \tag{5.16}$$

as an upper-bound approximation to V. Since for all $0 \leq S \leq R$

$$\underset{\sim}{V}(S) \leq V(S) \leq \tilde{V}(S)$$

one linearization serves to bound the approximation error to the other linearization.

In this section we consider methods to determine the sectoral supplier-equilibrium, when the infinite planning horizon is divided into two stages: a *transient stage* followed by a *stationary stage*. During the

transient stage the supplier's revenue function is given by

$$\beta^t(r) = \Phi^t(0) - \Phi^t(r) \qquad t=1,2,\ldots,T.$$

In the stationary stage,

$$\beta^t(r) = \Phi^{T+1}(0) - \Phi^{T+1}(r) \qquad t=T+1,T+2,\ldots.$$

Two possibilities will be considered. In the first case we assume that V(S) can be determined (or closely approximated) for all $0 \leq S \leq R$ by any applicable method. In the second case, approximations to V are updated by an iterative process. In the former approach, we separate the equilibrium sectoral-supplier's problem into two steps:

Step 1: Determine V for $0 \leq S \leq R$ for the stationary stage by solving

$$V(S) = \max_{0 \leq r \leq S} \{\Phi^{T+1}(0) - \Phi^{T+1}(r) - g(R-S,r) + \alpha V(S-r)\} \quad .$$

Step 2: Use V(S) as determined in step 1 to value stocks left at the end of the transient stage by solving for the finite-horizon sectoral supplier equilibrium (2.1) with salvage function

$$\beta^{T+1}(S) = V(S) \qquad \text{for all } 0 \leq S \leq R$$

In the second approach, the infinite-horizon supplier's problem is solved at each iteration as a finite-horizon supplier's problem with an additional piecewise-linear upper-bound approximation, to the salvage function, in other words

$$\tilde{\beta}^{T+1,n}(S) \equiv \tilde{V}^n(S) = \underset{k=1,2,\ldots K}{\text{minimum}} \{V^k + \pi^k(S-S^k)\}$$

for all $0 \leq S \leq R$. The upper-bound approximation $\tilde{V}$ is the outer linearization (5.16) of the concave function V.

At each iteration n, the finite-horizon supplier's problem (2.1), is solved using $\tilde{\beta}^{t,n}$, t=1,2,...,T+1. This will result in a supply schedule $(r^{1,n}, r^{2,n},\ldots,r^{T,n})$ and a stock left at the end of the transient stage, $S^{T+1,n}$. The supply schedule is then sent to the sectoral problems (2.10) or (2.16), t=1,2,...,T, and the stock left is sent to the stationary stage problem, to be evaluated. The sectoral problems are reoptimized at these levels, yielding costs $\Phi^t(r^{t,n})$ and shadow prices $\pi^{t,n+1}$, t=1,2,...,T as before. Similarly the stationary stage problem (5.1) (possibly by iterating with the stationary sectoral problem) is reoptimized at $S=S^{T+1,n}$ *yielding values* $V(S^{T+1,n})$ *and subgradients* $\pi^{T+1,n+1}$.

If

$$\tilde{\beta}^{t,n}(r^{t,n}) = \Phi^t(0) - \Phi^t(r^{t,n}) \qquad t=1,2,\ldots,T$$

and

$$\tilde{\beta}^{T+1,n}(S^{T+1,n}) = V(S^{T+1,n})$$

then an equilibrium has been reached. Conversely if either

$$\tilde{\beta}^{t,n}(r^{t,n}) > \Phi^t(0) - \Phi^t(r^{t,n})$$

for some t=1,2,...,T, or

$$\tilde{\beta}^{T+1,n}(S^{T+1,n}) > V(S^{T+1,n})$$

a tighter approximation is obtained for period t or for the stationary stage by adding to the approximations the new linear segment

$$\Phi^{t}(0) - \Phi^{t}(r^{t,n}) + \pi^{t,n+1}(r - r^{t,n})$$

or

$$V(S^{T+1,n}) + \pi^{T+1,n+1}(S - S^{T+1,n}) \quad .$$

The supplier's problem is then reoptimized at iteration n+1, using these improved approximations. The iterative approach is depicted in Figure 5.1.

This iterative approach is a further generalization of Geoffrion's [26] generalized Benders' iterative scheme. Consider for the case of convex sectoral problems the problem

$$\max \sum_{t=1}^{\infty} \alpha^{t-1} \{\Phi^{t}(0) - c^{t}(x^{t}) + f^{t}(s^{t}) - g^{t}\,(\sum_{j=1}^{t-1} r^{j}, r^{t})\}$$

$$\text{s.t.} \qquad F^{t}\,(d^{t};\; x^{t}, s^{t}, z^{t}\,) \leq 0 \qquad\qquad t=1,2,\ldots \qquad (5.17)$$

$$z^{t} \leq r^{t}$$

$$\sum_{t=1}^{\infty} r^{t} \leq R$$

$$0 \leq x^{t},\ 0 \leq s^{t},\ 0 \leq z^{t},\ 0 \leq r^{t} \qquad\qquad t=1,2,\ldots$$

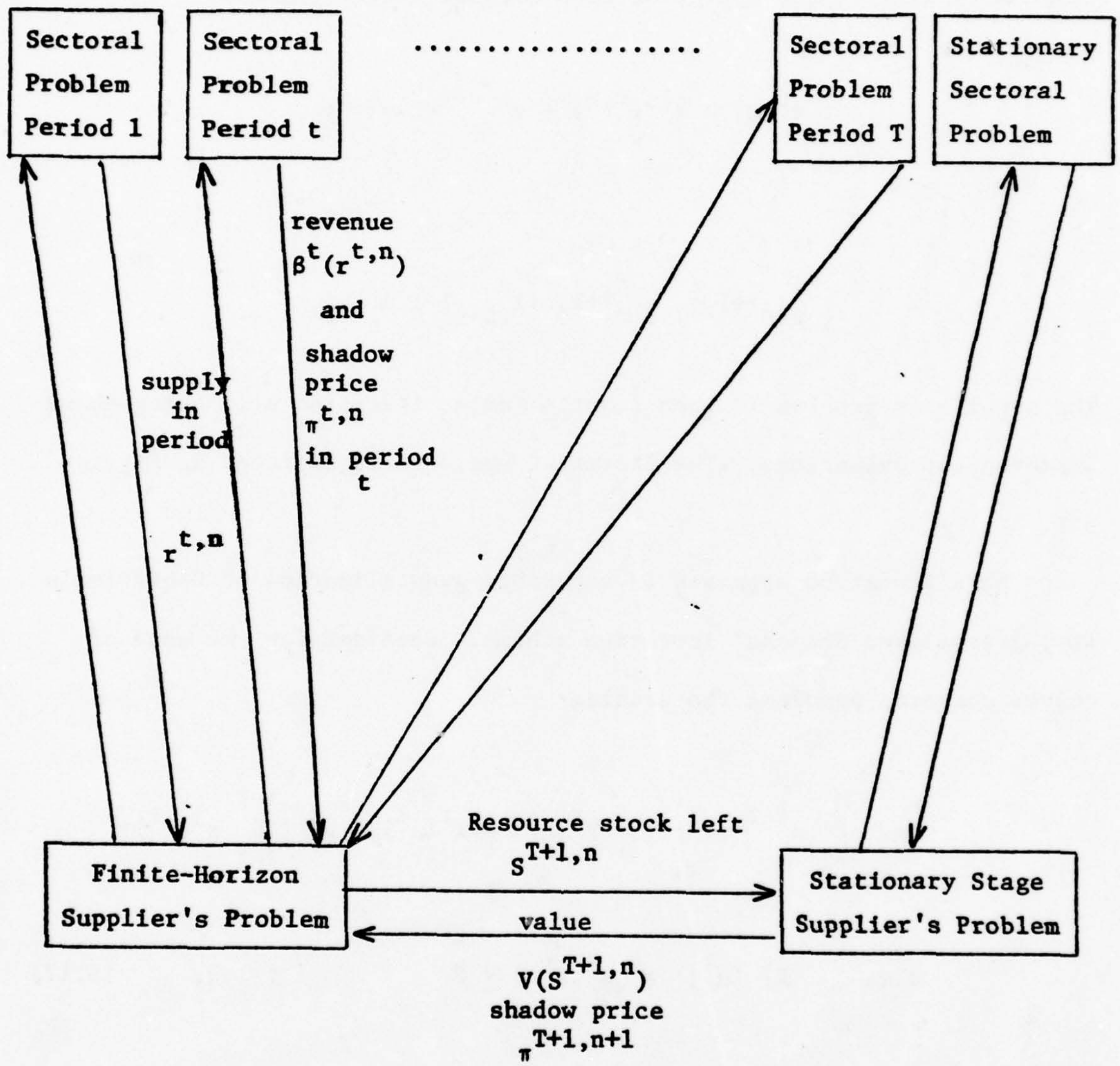

Sectoral Problem Period 1
Sectoral Problem Period t
....
Sectoral Problem Period T
Stationary Sectoral Problem
supply in period $r^{t,n}$
revenue $\beta^t(r^{t,n})$ and shadow price $\pi^{t,n}$ in period t
Finite-Horizon Supplier's Problem
Resource stock left $S^{T+1,n}$
value $V(S^{T+1,n})$ shadow price $\pi^{T+1,n+1}$
Stationary Stage Supplier's Problem

Figure 5.1

with stationarity holding after period T. Then for a fixed transient stage supply schedule $r^1, r^2, \ldots r^T$, problem (5.17) is separable into the T+1 subproblems, the T convex sectoral problems (2.16), and the stationary stage supplier's problem (5.1). The master problem is the finite-horizon supplier's problem (2.1) with

$$\beta^t(r) = \Phi^t(0) - \Phi^t(r) \qquad t=1,2,\ldots,T$$

and

$$\beta^{T+1}(S) = V(S) \quad .$$

Finally, we shall close this section by illustrating when the homogeneity properties of the sectoral problems can be explored to yield alternative specifications to stationarity after a certain period, to the infinite-horizon problem. We consider a simpler LP-sectoral problem of the form

$$\Phi^t(r) = \min \quad c^t x^t$$

$$\text{s.t.} \quad \rho^t x^t \leq r$$

$$A\, x^t \geq d^t$$

$$0 \leq x^t$$

Let

$$c_i^t = \delta^{t-1} c_i \qquad t=1,2,\ldots$$

$$d_j^t = \gamma^{t-1} d_j \qquad t=1,2,\ldots$$

and

$$\rho_k^t = \kappa^{t-1} \rho_k \qquad t=1,2,\ldots$$

for all i, j and k, where δ, γ and κ are respectively the geometric rates of cost decline, demand growth, and progress in resource efficiency utilization. It is easy to show by the linear homogeneity of linear programs that

$$\Phi^t(r) = (\delta\gamma)^{t-1}\Phi\left(\frac{r}{(\gamma\kappa)^{t-1}}\right) \qquad t=1,2,\ldots$$

Consider determining for $0 \leq S \leq R$ and $a,b \geq 0$ the function

$$V(S;a,b) = \max_{0\leq r\leq S} \{\alpha[\Phi(0) - \Phi[(\tfrac{r}{b})] - g(R-S,r) + \alpha V(S-r;\alpha\delta\gamma,b\gamma\kappa)\}$$

Starting at any initial state $0\leq S\leq R$, the resulting infinite-horizon supplier's problem is solved by computing

$$V(S;1,1) \quad .$$

5.IV. An Infinite-Horizon Linear Programming Model

In this section we consider solution procedures to an infinite-horizon linear program problem with depletable resources. The analysis here departs from that of previous chapters because we assume the depletable resource is owned and controlled by the economic sector. This assumption is convenient because it makes the interpretation of the solution procedure more tractable. However, it can be seen from the observation of (2.15) and (2.18) that for a depletable resource supplier receiving in each period the economic sector's cost savings equilibrium as defined by (2.9) is the solution to a problem equivalent to that faced by an economic sector that owns the resource stock acting such as to minimize the present value of the cost of meeting demand for end-use goods over the planning horizon.

The problem we consider in this section is

$$V(R) = \min \sum_{t=1}^{\infty} \alpha^{t-1} \{cx^t + g\,[z^t - z^{t-1}]\}$$

$$\begin{aligned} \text{s.t.} \quad & Ax^t = d^t & & t=1,2,\ldots \\ & z^t = z^{t-1} + \rho x^t & & t=1,2,\ldots \\ & 0 \le x^t \quad t=1,2,\ldots, \quad 0 \le z^t \le R & & t=1,2,\ldots \end{aligned} \tag{5.18}$$

where g denotes the extraction cost per unit, under the assumption of constant marginal extraction cost and z^t stands for the amount extracted by the end of period t, with $z^{\circ} = 0$.

<u>Definition 5.1</u>: A matrix A is Leontief, denoted $A \varepsilon L$ if each column of A

contains exactly one positive element and A transform some non-negative vector ($x \geq 0$) into a strictly positive vector ($Ax > 0$).

Facts about Leontief substitution systems that will be used here are summarized below.

<u>Proposition 5.1</u>: (i) If $A \epsilon L$ $d' > 0$, and the optimum of min cx s.t. $Ax = d'$, $x \geq 0$ exists, then the optimal basis remain optimal for any $d \geq 0$.

(ii) If $B\epsilon$ is square, $B^{-1} \geq 0$

Proof: See Koehler, et al. [53]. ||

The following assumptions are made for problem (5.18):

(i) $c \geq 0$, $g \geq 0$, $\rho \geq 0$, $d^t > 0$, $\alpha < 1$

(ii) $A \epsilon L$

(iii) the positive sequence of vectors of end-use demands $\{d^t\}$ is α-convergent or, in other words

$$d = \sum_{t=1}^{\infty} \alpha^{t-1} d^t < \infty$$

Consider the finite linear programming *auxiliary problem:*

$$\begin{aligned} \underset{\sim}{V}(R) &= \min\ cx + (1-\alpha)gz \\ \text{s.t.}\quad Ax &= d \\ -\rho x + (1-\alpha)\, z &= 0 \qquad\qquad (5.19) \\ 0 \leq x,\ 0 \leq z &\leq \frac{R}{1-\alpha} \end{aligned}$$

The importance of the auxiliary problem (5.19) stems from the possibility of obtaining relevant information for the solution of the original problem (5.18). As we shall see in some special cases solution to (5.18) can be obtained directly from solving (5.19). For this purpose, we show

Lemma 5.8: If the pair $\{x^t, z^t\}$ is feasible for the original problem (5.18) and α-convergent then the pair

$$\hat{x} = \sum_{t=1}^{\infty} \alpha^{t-1} x^t \geq 0$$

$$\hat{z} = \sum_{t=1}^{\infty} \alpha^{t-1} z^t \geq 0$$

is feasible for the auxiliary problem (5.19).

Proof: This follows trivially by showing

$$A\hat{x} = A \sum_{t=1}^{\infty} \alpha^{t-1} x^t = \sum_{t=1}^{\infty} \alpha^{t-1} Ax^t = \sum_{t=1}^{\infty} \alpha^{t-1} d^t = d$$

$$(1-\alpha)\hat{z} - \rho\hat{x} = (1-\alpha) \sum_{t=1}^{\infty} \alpha^{t-1} z^t - \rho \sum_{t=1}^{\infty} \alpha^{t-1} x^t = \sum_{t=1}^{\infty} \alpha^{t-1}(z^t - z^{t-1} - \rho x^t) = 0$$

$$\hat{z} = \sum_{t=1}^{\infty} \alpha^{t-1} z^t \leq \sum_{t=1}^{\infty} \alpha^{t-1} R = \frac{R}{1-\alpha} \quad . \|$$

Lemma 5.9: Every feasible solution $\{x^t, z^t\}$ to (5.18) is α-convergent.

Proof: This follows by observing that $0 \leq z^t \leq R$ implies that the sequence $\{z^t\}$ is α-convergent from Grinold's[34] result that for any matrix $A \epsilon L$ and

$$A\,x^t = d^t$$

there exists a $\kappa > 0$, such that

$$||x^t|| \leq \kappa\,||d^t||$$

Since $\{d^t\}$ is assumed α-convergent, the sequence $\{x^t\}$ is α-convergent. ||

Lemma 5.10: The optimal value of the auxiliary problem (5.19) is a lower bound to the optimal value of the original problem (5.18),

$$\underset{\sim}{V}(R) \leq V(R) .$$

Proof: From Lemmas 5.8 and 5.9, we observe that for every feasible solution for (5.18), there is a corresponding feasible solution of (5.8). Furthermore these solutions have the same objective value since

$$c\hat{x} + (1-\alpha)\,gz = \sum_{t=1}^{\infty} \alpha^{t-1}\,\{cx^t + g[z^t - z^{t-1}]\} .$$

Consequently $\underset{\sim}{V}(R) \leq V(R)$. ||

Concerning the implications of the auxiliary problem (5.19) for solutions to the original problem (5.18), we need observe that

$$\begin{bmatrix} A & 0 \\ -\rho & 1-\alpha \end{bmatrix} \varepsilon\, L$$

Partitioning the matrix A into $A = [A_1, A_2]$ where A_1 corresponds to the columns of A for which $\rho_i = 0$ and A_2 to those for which $\rho_i > 0$, we have

<u>Theorem 5.3</u>: (i) If the auxiliary problem (5.19) is infeasible then the original problem (5.18) is infeasible.

(ii) If $A_1 \varepsilon L$ and in an optimal solution to the auxiliary problem (5.19) $\hat{z} = 0$, then

$$\begin{cases} x_{1B}^t = B_1^{-1} d^t & t=1,2,\ldots \\ x_{1N}^t = 0 & t=1,2,\ldots \\ x_2^t = 0 & t=1,2,\ldots \\ z^t = 0 & t=1,2,\ldots \end{cases}$$

where B_1 is a matrix composed of the columns of A_1 in the optimal basis, is an optimal solution to the original problem (5.18).

(iii) If in an optimal solution to the auxiliary problem (5.19),

$$0 < \hat{z} < \frac{R}{1-\alpha} \quad \text{and} \quad \sum_{t=1}^{\infty} \rho_B B^{-1} d^t \leq R$$

where B is a matrix composed of the columns of A in the optimal basis, then

$$\begin{cases} x_B^t = B^{-1} d^t & t=1,2,\ldots \\ x_N^t = 0 & t=1,2,\ldots \\ z^t = \sum_{j=1}^{t} \rho_B B^{-1} d^j & t=1,2,\ldots \end{cases}$$

is an optimal solution to the original problem (5.18).

Proof: The proof of (i) is carried by contradiction. Suppose the auxiliary problem (5.19) is infeasible, but the original problem (5.18) is feasible. Then by Lemmas 5.8 and 5.9 to any feasible solution to the original problem there corresponds a feasible solution to the auxiliary problem, leading to the desired contradiction.

The proof of (ii) follows from observing that if $\hat{z} = 0$ in (5.19) then $\rho\hat{x} = 0$. Partitioning correspondingly to $A-[A_1,A_2]$, $x' = [x_1',x_2']$, $c = [c_1,c_2]$ and $\rho = [0,\rho_2]$ then $x = [\hat{x}_1,0]$ and $\hat{x}_1$ is optimal for

$$\min\ c_1x_1$$

$$\text{s.t.}\quad A_1x_1 = d$$

$$0 \leq x_1$$

Letting $A_1 = [B_1,N_1]$, where B_1 corresponds to the columns for which $\hat{x}_1 > 0$, then $B^{-1} \geq 0$ since $A_1 \varepsilon L$ by assumption and Proposition 5.1. Therefore

$$\underset{\sim}{V}(R) = c_{1B}\, B_1^{-1}\, d$$

Consequently

$$\left[\begin{array}{ll}
\hat{x}_{1B}^t = B_1^{-1} d^t \geq 0 & t=1,2,\ldots \\
\hat{x}_{1N}^t = 0,\ \hat{x}_2^t = 0 \text{ and } \hat{z}^t = 0 & t=1,2,\ldots \\
B_1\hat{x}_{1B}^t + N_1\hat{x}_{1N}^t + A_2\hat{x}_2^t = d^t & t=1,2,\ldots \\
\hat{z}^t = \hat{z}^{t-1} + \rho_2\hat{x}_2^t = z^{t-1} = 0 & t=1,2,\ldots
\end{array}\right.$$

is feasible to the original problem with objective value

$$\sum_{t=1}^{\infty} \alpha^{t-1} c_{1B} B_1^{-1} d^t = c_{1B} B_1^{-1} d = \underset{\sim}{V}(R)$$

From Lemma 5.10, this solution attains the lower bound and therefore is optimal for the original problem (5.18). This completes the proof of (ii).

For the proof of (iii), since $0 < \hat{z} < \frac{R}{1-\alpha}$ we denote the optimal basis for the auxiliary problem by

$$\begin{bmatrix} B & 0 \\ -\rho_B & 1-\alpha \end{bmatrix} \quad \text{with inverse} \quad \begin{bmatrix} B^{-1} & 0 \\ \dfrac{\rho_B B^{-1}}{1-\alpha} & \dfrac{1}{1-\alpha} \end{bmatrix} \geq 0$$

and its optimal solution by

$$\begin{cases} \hat{x} = B^{-1} d \\ \hat{z} = \dfrac{\rho_B B^{-1} d}{1-\alpha} \end{cases}$$

with value $\underset{\sim}{V}(R) = c_B B^{-1} d + g \rho_B B^{-1} d$

The solution

$$\begin{cases} \hat{x}_B^t = B^{-1} d^t \geq 0 & t=1,2,\ldots \\ \hat{x}_N^t = 0 & t=1,2,\ldots \\ \hat{z}^t = \sum_{j=1}^{t} \rho_B B^{-1} d^j \geq 0 & t=1,2,\ldots \end{cases}$$

is feasible for the original problem (5.18) if

$$\hat{z}^t = \sum_{j=1}^{t} \rho_B B^{-1} d^j \leq \sum_{j=1}^{\infty} \rho_B B^{-1} d^j \leq R$$

because

$$B \hat{x}_B^t + N \hat{x}_N^t = d^t \qquad t=1,2,\ldots$$

$$\hat{z}^t = \sum_{j=1}^{t} \rho_B B^{-1} d^j = \sum_{j=1}^{t-1} \rho_B B^{-1} d^j + \rho_B B^{-1} d^t$$

$$= \hat{z}^{t-1} + \rho_B \hat{x}_B^t + \rho_N \hat{x}_N^t$$

The value of the objective for this feasible solution is

$$\sum_{t=1}^{\infty} \alpha^{t-1} [c_B B^{-1} d^t + g \rho_B B^{-1} d^t] = c_B B^{-1} d + g \rho_B B^{-1} d = \underset{\sim}{V}(R)$$

Since this solution is feasible to (5.18) and attains the lower bound given by Lemma 5.10, it is optimal for the original problem (5.18).

For the cases not covered by Theorem 5.3, an approximation method is suggested later in this section. Notice that Theorem 5.3 covers an ensemble of special cases from which the solution to the original problem (5.18) is obtained directly from the solution to the auxiliary problem (5.19). Apparently these are the only cases for which the solution to (5.18) is obtained at a stationary vertex of A, given by [B,B,....]. A comparison between these results and those of Grinold [35] and Grinold and Hopkins [36] leads to the immediate conclusion that the existence of a

finite stock of a depletable resource represented by means of upper-bound constraints in the auxiliary problem (5.19), will in general destroy the Leontief properties that lead to stationary vertex solutions.

For instance for the case covered by Theorem 5.3(iii), a necessary condition for the series

$$\sum_{t=1}^{\infty} \rho_B B^{-1} d^t$$

to converge is that $\lim_{t\to\infty} d^t = 0$

which makes this case in general not realistic. A finite stock of a depletable resource cannot support a solution at a stationary vertex that consumes the resource positively, or in other words, satisfying

$$\rho_B B^{-1} d^t > 0$$

if the end-use demands are either growing or stationary, because in the latter case we will have

$$\lim_{T\to\infty} \sum_{t=1}^{T} \rho_B B^{-1} d^t = \infty .$$

However, the solution to the auxiliary problem (5.19) is felt to be insightful at least from the fact that it provides by Lemma 5.10 lower bounds to the value of the original problem, relevant for the evaluation of approximating solutions.

We now suggest an approximation method for (5.18), assuming that $A_1 \varepsilon L$ and that

$$\min c_1 x_1$$
$$\text{s.t.} \quad A_1 x_1 = d \qquad (5.20)$$
$$0 \le x_1$$

has a solution for some positive right-hand side ($d' > 0$). From Proposition 5.1, the optimal basis B_1 remains optimal for all non-negative right-hand sides ($d^t \ge 0$). Therefore the value of the optimal solution to (5.20) is for all $d^t \ge 0$, ud^t, where $u = c_{1B}B_1^{-1}$.

Let $d_T = \sum_{t=T}^{\infty} \alpha^{t-T} d^t$, which is finite because the sequence $\{d^t\}$ is assumed α-convergent, and consider the *T-period approximation*

$$\tilde{V}_T(R) = \alpha^T u d_{T+1} + \min \sum_{t=1}^{T} \alpha^{t-1} \{cx^t + g[z^t - z^{t-1}]\}$$
$$\text{s.t.} \qquad Ax^t = d^t \qquad t=1,2,\dots,T$$
$$(5.21)$$
$$-\rho x^t + z^t - z^{t-1} = 0 \quad t=1,2,\dots,T$$
$$0 \le x^t,\ 0 \le z^t \le R \qquad t=1,2,\dots,T$$

We can then show

Lemma 5.11: $\tilde{V}_T(R) \geq \tilde{V}_{T+1}(R) \geq V(R)$

Proof: We show first that $\tilde{V}_T(R) \geq V(R)$. Consider further restricting the original problem (5.18) to satisfy $z^t = z^T$ for $t \geq T+1$. It is then easy to see that the activities involving positive depletable resource usage cannot be operated for $t \geq T+1$ ($x_2^t = 0$, $t = T+1, T+2, \ldots$). The original problem (5.18) is then separable into

$$\min \sum_{t=1}^{T} \alpha^{t-1} \{cx^t + g\,[z^t - z^{t-1}]\}$$

$$\text{s.t.} \quad Ax^t = d^t \qquad t=1,2,\ldots,T$$

$$-\rho x^t + z^t - z^{t-1} = 0 \qquad t=1,2,\ldots,T$$

$$0 \leq x^t,\ 0 \leq z^t \leq R$$

for the $t = T+1, T+2, \ldots$ into the problems (5.20)

$$ud^t = \min c_1 x_1^t$$

$$\text{s.t.} \quad A_1 x_1^t = d^t$$

$$0 \leq x_1^t$$

We may conclude then that

$$\tilde{V}_T(R) \geq V(R) \ .$$

To show that $\tilde{V}_T(R) \geq \tilde{V}_{T+1}(R)$, it suffices to observe that the T-period approximation (5.21) can be obtained from the T+1-period approximation, by further restricting it to satisfy

$$z_{T+1}^{T} = z_{T+1}^{T+1} \quad . \quad ||$$

The above approximation method is indeed a search for the end-of the transient stage, after which the depletable resource is exhausted and operation can only continue on the basis of activities that do not consume any of the depletable resource.

General convergence of the T-period approximations (5.21) that would guarantee that

$$\lim_{T\to\infty} \tilde{V}_T(R) = V(R)$$

will not be proved here. However it is felt that it should not seriously depart from the proofs of Theorems (5.1) and (5.2) exploring the contraction imposed by $\alpha < 1$. However we do show a companion result,

Lemma 5.12: If in the solution to the original problem, $z^t = z^T$ $t \geq T+1$, for some period T, then the T-period approximations (5.21) converge after at most T approximations.

Proof: This follows by showing that $z_T^t = z^t$, t=1,2,...,T, is feasible for the T-period approximation with an objective value equal to V(R). Since this feasible solution attains a lower bound, as a result of Lemma 5.11, it is optimal for the T-period approximation and

$$\tilde{V}_T(R) = V(R) \quad . \quad ||$$

A few numerical examples will illustrate the results of this section.

<u>Example 5.1</u>: Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad d^t = \begin{bmatrix} 2^{-t+2} \\ 2^{-t+2} \end{bmatrix} \qquad t=1,2,\ldots$$

$$c = [1 \quad 2 \quad 5] \qquad \rho = [0 \quad 0 \quad 1/2]$$

$\alpha = 1/2$, $g = 2$ and $R = 2$.

The auxiliary problem (5.19) is then

$$\underset{\sim}{V}(2) = \min x_1 + 2x_2 + 5x_3 + z$$

$$\text{s.t.}$$

$$x_1 - x_2 = 8/3$$

$$x_2 + x_3 = 8/3$$

$$-1/2x_3 + 1/2z = 0$$

$$0 \leq x_1, x_2, x_3 \quad \text{and} \quad 0 \leq z \leq 4$$

with optimal solution $\hat{x}_1 = 16/3$, $\hat{x}_2 = 8/3$ and $\hat{x}_3, \hat{z} = 0$. From Theorem 5.3 (ii), with

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

for t=1,2,...

$$\begin{bmatrix} x_1^t \\ \\ x_2^t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{-t+2} \\ \\ 2^{-t+2} \end{bmatrix} = \begin{bmatrix} 2^{-t+3} \\ \\ 2^{-t+3} \end{bmatrix}$$

and $x_3^t, z^t = 0$ is an optimal solution to the original problem.

Example 5.2: Let in Example 5.1

$$c = [1 \quad 2 \quad 0]$$

instead. The auxiliary (5.19) is then

$$\underset{\sim}{V}(2) = \min x_1 + 2x_2 + z$$

s.t.

$$x_1 - x_2 \qquad\qquad = 8/3$$

$$x_2 + x_3 \qquad = 8/3$$

$$-1/2\, x_3 + 1/2z = 0$$

$$0 \leq x_1, x_2, x_3 \quad \text{and} \quad 0 \leq z \leq 4$$

with optimal solution $\hat{x}_1 = 8/3$, $\hat{x}_3 = 8/3$, $\hat{z} = 8/3$ and $\hat{x}_2 = 0$ and basis

$$B = B^{-1} = \begin{bmatrix} 1 & 0 \\ \\ 0 & 1 \end{bmatrix}$$

Furthermore since for t=1,2,...

$$\rho_B B^{-1} d^t = [0 \quad 1/2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{-t+2} \\ 2^{-t+2} \end{bmatrix} = 2^{-t+1}$$

and consequently

$$\sum_{t=1}^{\infty} 2^{-t+1} = 2 = R,$$

by Theorem 5.3 (iii), the solution

$$\begin{bmatrix} x_1^t \\ x_3^t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{-t+2} \\ 2^{-t+2} \end{bmatrix} = \begin{bmatrix} 2^{-t+2} \\ 2^{-t+2} \end{bmatrix} \quad t=1,2,...$$

$$x_2^t = 0 \quad \text{and} \quad z^t = \sum_{i=1}^{t} 2^{-i+1}$$

is optimal for the original problem.

This example illustrates a case where the depletable resource is exhausted in infinite time.

<u>Example 5.3</u>: Let in Example 5.1,

$$d^t = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad c = [1 \quad 2 \quad 0]$$

The auxiliary problem (5.19) becomes then

$$\underset{\sim}{V}(2) = \min \; x_1 + 2x_2 \qquad\qquad + z$$

$$\text{s.t. } x_1 - x_2 \qquad\qquad = 2$$

$$x_2 + x_3 \qquad = 2$$

$$-1/2x_3 + 1/2z = 0$$

$$0 \le x_1, x_2, x_3 \quad \text{and} \quad 0 \le z \le 4 \quad .$$

with optimal solution $\hat{x}_1=2$, $\hat{x}_3=2$, $\hat{z}=2$, $\hat{x}_2=0$, $\tilde{V}(2)=4$, and basis

$$B = B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since

$$\rho_B B^{-1} d^t = [0 \quad 1/2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1/2$$

we have

$$\sum_{t=1}^{\infty} \rho_B B^{-1} d^t = +\infty \quad .$$

Apparently the optimal solution to the original problem is

$$\begin{cases} x_1^t = 1 \\ x_2^t = 0 \\ x_3^t = 1 \qquad \text{for } t=1,2,3,4 \text{ and} \\ z^t = \sum_{t=1}^{t} 1/2 \end{cases} \qquad \begin{cases} x_1^t = 2 \\ x_2^t = 1 \\ x_3^t = 0 \qquad \text{for } t=5,6,\ldots \\ z^t = 2 \end{cases}$$

The objective value of this solution is 17/4 not very far from the lower bound given by the auxiliary problem ($\underset{\sim}{V}(2) = 4$)).

5.V. An Ascent Algorithm for Infinite-Horizon Problems with Discounting

In this section we digress from the framework of the sectoral-supplier equilibrium to consider a class of problems likely to arise in conjunction with the solution of general infinite-horizon programs with discounting. These are in general the "dual" problems proposed by Grinold [31,32] for problem (5.11) and Evers [18,19]. For this purpose we need the following definitions

<u>Definition 5.2</u>: Let p be a real number, $1 \leq p < \infty$. The space ℓ_p denotes all sequences of scalars $\{u\}$ for which

$$\sum_{t=0}^{\infty} |u^t|^p < +\infty$$

with norm

$$||u||_p = \left(\sum_{t=0}^{\infty} |u^t|^p \right)^{1/p}$$

and metric

$$||u_1 - u_2||_p = \left(\sum_{t=0}^{\infty} |u_1^t - u_2^t|^p \right)^{1/p}$$

<u>Definition 5.3</u>: The space ℓ_∞ consists of all sequence of scalars $\{u\}$ such that

$$\sup |u^t| < +\infty$$

with norm

$$||u||_\infty = \sup |u^t|$$

and metric

$$||u_1 - u_2||_\infty = \sup |u_1^t - u_2^t|$$

We introduce the notation

$$\{u_\alpha\} = \{u^0, \alpha u^1, \alpha^2 u^2, \ldots\}$$

Proposition 5.2: $\ell_p \quad \ell_{p'}$, for $p \leq p'$

Proof: See Fomin [24]. ||

Lemma 5.13: If $\{u_{1/\alpha}\} \varepsilon \ell_\infty$ then $\{u\} \varepsilon \ell_1$

Proof: If $\sup \left|\frac{u^t}{\alpha^t}\right| < M$ then $|u^t| < \alpha^t M$, $t=0,1,2$ and therefore

$$\sum_{t=0}^{\infty} |u^t| < \frac{M}{1-\alpha} \quad .$$

Consequently $\{u\} \varepsilon \ell_1$. ||

Lemma 5.14: If $\{\gamma\} \varepsilon \ell_\infty$ then $\{\gamma_\alpha\} \varepsilon \ell_1$

Proof: If $\sup |\gamma^t| < M$ then $|\gamma^t| < M, t=0,1,\ldots$, which after multiplication by α^t yields

$$\sum_{t=0}^{\infty} |\alpha^t \gamma^t| < \frac{M}{1-\alpha} \quad .$$

Hence

$$\{\gamma_\alpha\} \varepsilon \ell_1 \ . \ ||$$

Lemma 5.15: If $\{\gamma\} \varepsilon \ell_\infty$ and $\{u\} \varepsilon \ell_1$, then

$$\sum_{t=0}^{\infty} u^t \gamma^t < + \infty$$

Proof: Follows from Holder's inequality (see Luenberger [57])

$$\left|\sum_{t=0}^{\infty} u^t \gamma^t\right| < ||u||_1 \, ||\gamma||_\infty \, . ||$$

Consider the following optimization problem

$$\max L(u)$$

$$\text{s.t. } \{u_{1/\alpha}\} \in \ell_\infty \qquad (5.22)$$

where L is continuous, concave, real-valued and finite for all sequences $\{u_{1/\alpha}\} \in \ell_\infty$ where $\alpha < 1$. We assume that problem (5.22) has an optimal solution $\hat{u}$.

The generalization of the concept of the subgradient for functions defined in an infinite-dimensional space would say that a sequence $\{\bar{\gamma}\}$ is a subgradient of L at $\bar{u}$ if

$$L(u) \leq L(\bar{u}) + (u-\bar{u})\,\bar{\gamma} \quad \text{for all } u. \qquad (5.23)$$

The ascent algorithm is motivated by

Lemma 5.16: The set $\{u \mid (u-\bar{u})\,\bar{\gamma} \geq 0\}$ contains all optimal solutions to (5.22).

Proof: From (5.23), for an optimal u

$$(u-\bar{u})\,\bar{\gamma} \geq L(u) - L(\bar{u}) \geq 0 \quad . ||$$

It seems plausible then to construct ascent algorithms for (5.22) of the form

$$u^{t,n+1} = u^{t,n} + \alpha^t \, \theta \;\; \gamma^t \qquad t=0,1,2,\ldots \qquad (5.24)$$

where θ is a scalar. We can then show

Lemma 5.17: If $\{u^n_{1/\alpha}\} \in \ell_\infty$, $0 \leq \theta \leq \bar{\theta}$, and $\{\gamma\} \in \ell_\infty$ then

$$\{u^{n+1}_{1/\alpha}\} \in \ell_\infty$$

Proof: This result follows from dividing (5.23) by α^t and applying the triangle inequality

$$\sup \left| \frac{u^{t,n+1}}{\alpha^t} \right| \leq \sup \left| \frac{u^{t,n}}{\alpha^t} \right| + \bar{\theta} \sup |\gamma^t| \ .$$

Therefore

$$||u^{n+1}_{1/\alpha}||_\infty = \sup \left| \frac{u^{t,n+1}}{\alpha^t} \right| < + \infty$$

and consequently

$$\{u^{n+1}_{1/\alpha}\} \in \ell_\infty \ . \ ||$$

Theorem 5.4: Consider the problem (5.22) with an optimal solution $\hat{u}$. Suppose we apply subgradient optimization (5.24) with the additional assumption that $||\gamma||_\infty < M$ and step length

$$\theta = \frac{\lambda[L(\hat{u}) - L(u^n)]}{||\gamma^n_\alpha||^2_2}$$

with $0 < \varepsilon_1 \leq \lambda \leq 2-\varepsilon_2$ and $\varepsilon_2 > 0$, then

$$\lim_{n\to\infty} L(u^n) = L(\hat{u}) \quad .$$

Proof: First we show $||u^n - \hat{u}||_2^2$ is decreasing, or in other words, that the distance from u^n to $\hat{u}$ in the ℓ_2 space, is decreasing

$$||u^{n+1} - \hat{u}||_2^2 = ||u^n + \theta^n \gamma_\alpha^n - \hat{u}||_2^2$$

$$= ||u^n - \hat{u}||_2^2 + (\theta^n)^2 \; ||\gamma_\alpha^n||^2 + 2\,\theta^n (u^n - \hat{u})\,\gamma_\alpha^n$$

Now since $\alpha < 1$ and therefore

$$\sum_{t=0}^{\infty} \alpha^t (u^{t,n} - \hat{u}^t)\,\gamma^{t,n} < \sum_{t=0}^{\infty} (u^{t,n} - \hat{u}^t)\,\gamma^{t,n}$$

we have

$$||u^{n+1} - \hat{u}||_2^2 < ||u^n - \hat{u}||_2^2 + (\theta^n)^2 \; ||\gamma_\alpha^n||_2^2 + 2\theta^n (u^n - \hat{u})\,\gamma^n \qquad (5.25)$$

But since

$$0 \leq L(\hat{u}) - L(u^n) \leq (\hat{u} - u^n)\,\gamma^n$$

$$(u^n - \hat{u})\,\gamma^n \leq L(u^n) - L(\hat{u})$$

we have in (5.25) that

$$||u^{n+1} - \hat{u}||_2^2 < ||u^n - \hat{u}||_2^2 + (\theta^n)^2 \; ||\gamma_\alpha^n||_2^2 - 2\,\theta^n\,[L(\hat{u}) - L(u^n)]$$

Substituting for θ yields

$$||u^{n+1} - \hat{u}||_2^2 < ||u^n - \hat{u}||_2^2 + (\lambda^2 - 2\lambda)\,\frac{[L(\hat{u}) - L(u^n)]^2}{||\gamma_\alpha^n||_2^2}$$

Let now

$$\lambda^2 - 2\lambda = \lambda\ (\lambda - 2)$$

and for

$$\lambda \geq \varepsilon_1 > 0$$

$$\lambda - 2 \leq \varepsilon_2 \quad \text{and} \quad \varepsilon_2 > 0,$$

we have

$$\lambda(\lambda - 2) \leq - \varepsilon_2\, \varepsilon_1 < 0.$$

Consequently

$$||u^{n+1} - \hat{u}||_2^2 < ||u^n - \hat{u}||_2^2$$

The result is that the sequence of non-negative number $||u^n - \hat{u}||_2^2$ is decreasing and bounded from below. Hence

$$\lim_{n\to\infty} ||u^n - \hat{u}||_2^2 = 0 .$$

This implies that we must have

$$\lim_{n\to\infty} \frac{[L(\hat{u}) - L(u^n)]^2}{||\gamma_\alpha^n||_2^2} = 0$$

since otherwise there would be an infinite subsequence of $||u^{n+1} - \hat{u}||_2^2$ decreasing at each step by at least $\varepsilon_3 > 0$ which is clearly impossible. Since $||\gamma_\alpha^n||_2^2 < \infty$ because we assumed $\{\gamma\} \in \ell_\infty$ and of Lemma 5.14 we must have that

$$\lim_{n\to\infty} L(u^n) = L(\hat{u}) \quad . \ ||$$

The above extension to infinite dimensions of the finite-dimensional subgradient optimization algorithm (see Held and Karp [39], Held, Wolfe and Crowder [40]) has several practical difficulties. The first difficulty stems from the infinite dimension of the subgradients, for which finite approximation methods need be devised. Second, there is the mathematical assumption that ℓ_∞-norms of the subgradients are bounded. The next difficulty is that in general we do not know the value $L(\hat{u})$ of the maximal objective function in (5.22) to determine the step length. Therefore it need be estimated. Given these difficulties, the theoretical convergence cannot be guaranteed in practice.

Chapter 6: Equilibrium Generalized and Further Extensions

6.I. Introduction

In this chapter we discuss some further extensions in the framework of an equilibrium between a depletable resource supplier and an economic sector that is the sole user of this resource. Research in some of the topics discussed have been carried to different extents, justifying a separate treatment in the following sections.

6.II. A General Equilibrium Model

So far in the analysis of the previous chapters we assumed that the demand for the sectoral end-use goods was price-inelastic in all time periods in the planning horizon. The depletable resource supplier(s) revenue functions were then derived from the sectoral cost-savings in meeting given demands for end-use goods in each time period. In this section, we consider a model in which both the end-use demands, d^t, and the alternative primary supplies, s^t, are endogenous and price-elastic. What will differentiate in this model the depletable resource allocation from that of the other endogenous resources is its normative treatment. In the spirit of Manne's [61] E.T.A. model (see section 1.II), the cost of meeting demand will result from a "sectoral" problem which simultaneously minimizes two cost elements: operational and substitution costs. Alternatively the objective can be interpreted as the maximization of surplus: end-use goods consumers' surplus, alternative primary supplies producers' surplus

and the sectoral benefit from expenditures in other items out of a fixed budget, under the assumption of constant marginal utility.

The "sectoral" problems generating the cost function, ϕ^t, that we analyze here, are equivalent to the problems studied by Shapiro [79] from whom much of the discussion on the relationship between mathematical programming and economic equilibria models is inherited. His approach is extended here to a finite planning horizon and links together, in each period: end-use goods and alternative primary supplies econometric submodels with linear programming models of the economic sector and a depletable resource mathematical programming intertemporal supply model. This will generate an equilibrium in two levels: 1) between the economic sector and the end-use goods consumers and the alternative primary supplies producers; 2) between the economic sector and the depletable resource supplier(s). A two-level decomposition scheme can be envisaged to find a general equilibrium solution. This is depicted in Figure 6.1.

The most general type of "sectoral" problems that we will consider here to compute in each period the economic sector's cost of meeting demand for end-use goods is given by the *market-sectoral problem*

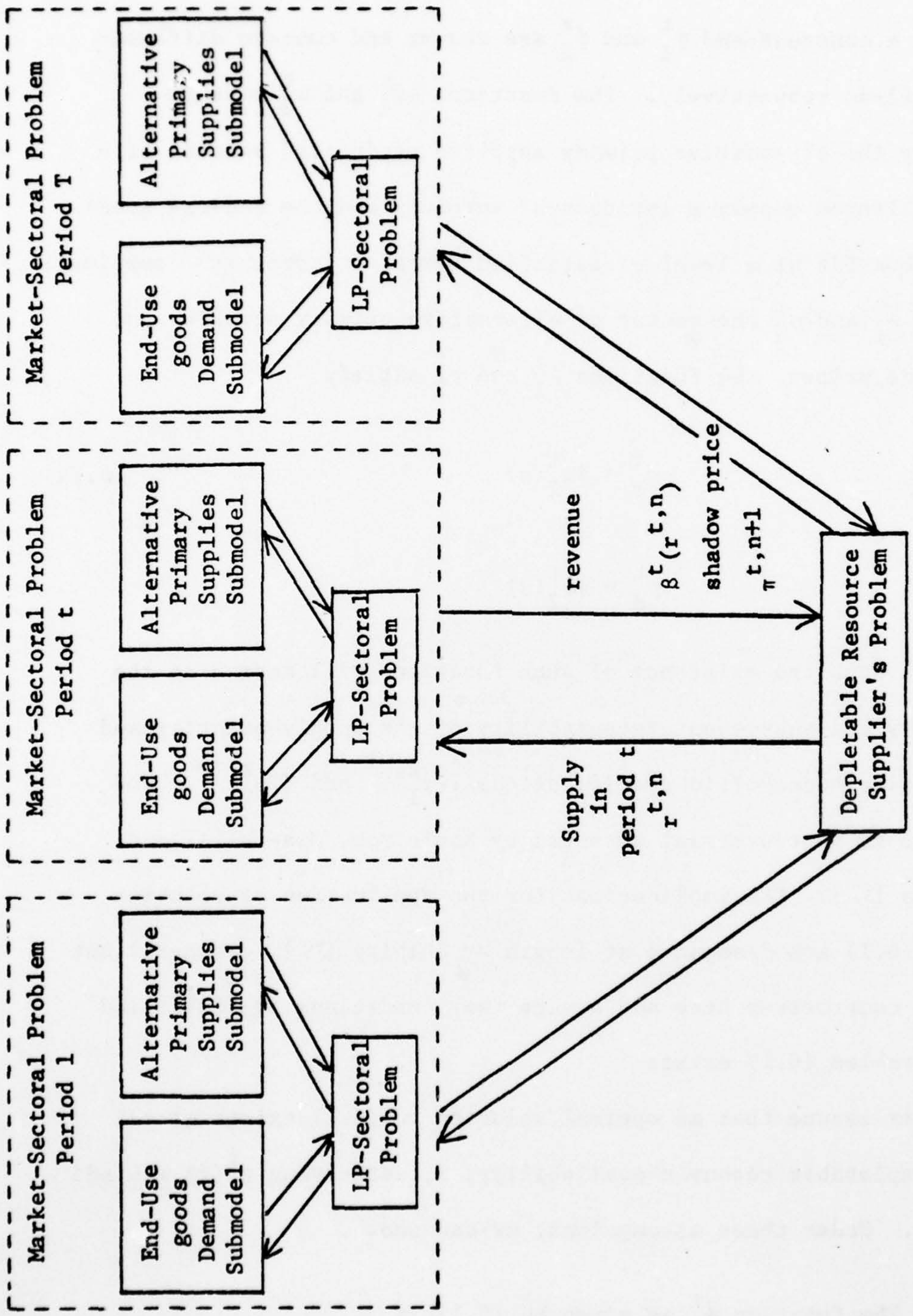
$$\phi^t(r) = K^t + \min c^t x^t + \psi_1^t(s) - \psi_2^t(d) \tag{6.1a}$$

$$\text{s.t} \quad \rho^t x^t \leq r \tag{6.1b}$$

$$A_1^t x^t \quad -s \quad \leq 0 \tag{6.1c}$$

$$A_2^t x^t \quad -d \geq 0 \tag{6.1d}$$

$$0 \leq x^t \text{ , } 0 \leq s \quad \text{and} \quad 0 \leq d \tag{6.1e}$$



Market-Sectoral Problem Period 1
End-Use goods Demand Submodel
Alternative Primary Supplies Submodel
LP-Sectoral Problem
Market-Sectoral Problem Period t
End-Use goods Demand Submodel
Alternative Primary Supplies Submodel
LP-Sectoral Problem
Market-Sectoral Problem Period T
End-Use goods Demand Submodel
Alternative Primary Supplies Submodel
LP-Sectoral Problem
Supply in period t $r^{t,n}$
revenue $\beta^t(r^{t,n})$
shadow price $\pi^{t,n+1}$
Depletable Resource Supplier's Problem

Figure 6.1

where K^t is a constant and ψ_1^t and ψ_2^t are convex and concave differentiable functions respectively. The functions $-\psi_1^t$ and ψ_2^t measure respectively the alternative primary supplies producers' benefit at a level of delivered supply s (producers' surplus) and the end-use goods consumers' benefit at a level of satisfied demand d (consumers' surplus). Denoting by p_s^t and p_d^t the vector of alternative primary supplies and end-use goods prices, the functions ψ_1^t and ψ_2^t satisfy

$$p_s^t = \nabla\psi_1^t(s) \qquad (6.2a)$$

$$p_d^t = \nabla\psi_2^t(d)$$

In empirical work the existence of such functions will depend on the reversibility and subsequent integrability of the supply-quantity and demand-quantity econometric specifications $(\nabla\psi_1^t)^{-1}$ and $(\nabla\psi_2^t)^{-1}$. The latter issue is controversial as noted by Kihlstrom, Mas-Collel and Sonnerschein [51]. Its implications for the application of solution methods to (6.1) are discussed at length by Shapiro [79]. We shall not address the controversy here and assume that conditions are fulfilled such that problem (6.1) exists.

We further assume that an optimal solution to (6.1) exists at all levels of depletable resource availability, r, satisfying $\phi^t(r) \geq 0$ and $\phi^t(0) < +\infty$. Under these assumptions, we can show

<u>Lemma 6.1</u>: The function ϕ^t as given by (6.1) is

(i) non-increasing

(ii) convex.

Proof: The proof of (i) is trivial and similar to that of Lemmas 2.1 and 2.2. From the concavity of ψ_2^t and the convexity of ψ_1^t, the proof of part (ii) also follows trivially. ||

Letting π^t, w^t and u^t denote the vector of shadow prices on constraints (6.1b), (6.1c) and (6.1d) respectively and defining

$$\phi^t(r;s,d) = K^t + \psi_1^t(s) - \psi_2^t(d) + \min c^t x^t$$

$$\begin{aligned} \text{s.t.} \quad & \rho^t x^t \le r \\ & A_1^t x^t - s \le 0 \\ & A_2^t s^t - d \ge 0 \\ & 0 \le x^t \end{aligned}$$

then,

$$\phi^t(r) = \min_{\substack{s \ge 0 \\ d \ge 0}} \phi^t(r;s,d)$$

Linear programming duality theory gives, omitting the constant for convenience, the cost-savings function (2.7) as

$$\phi^t(0) - \phi^t(r) = \underset{\substack{s \ge 0 \\ d \ge 0}}{\text{maximum}} \left\{ -\psi_1^t(s) + \psi_2(d) + \underset{k=1,2,\ldots,K^t}{\text{minimum}} \{(\phi^t(0) - u^{t,k}d + w^{t,k}s) + \pi^{t,k}r)\} \right\}$$

where k indexes the extreme points of the dual problem

$$\max\ -\ \pi^t r\ -\ v^t s + u^t d$$

$$\text{s.t.}\ -\ \pi^t \rho^t - v^t A_1^t + u^t A_2^t \leq c^t \tag{6.4}$$

$$0 \leq \pi^t\ ,\ 0 \leq v^t \text{ and } 0 \leq u^t \tag{6.5}$$

The convex polyhedral set formed by (6.4) and (6.5) is assumed bounded for simplicity.

As mentioned earlier in this section the derivation of Φ^t can be seen in two manners. In the formulation (6.1), Φ^t is computed in each period by the minimization of

$$c^t x^t \quad + \quad [\psi_1^t(s) - \psi_1^t(\mathit{s})] \quad + \quad [\psi_2^t(\mathit{d}) - \psi_2^t(d)]$$

$$\begin{pmatrix}\text{sectoral operatio-}\\ \text{nal costs in}\\ \text{period } t\end{pmatrix} + \begin{pmatrix}\text{alternative primary}\\ \text{supplies' substitu-}\\ \text{tion costs in period } t\end{pmatrix} + \begin{pmatrix}\text{end-use goods}\\ \text{substitution costs}\\ \text{in period } t\end{pmatrix}$$

where s and d are arbitrarily set reference levels, in which case

$$K^t = \psi_2^t(\mathit{d}) - \psi_2^t(\mathit{s})$$

in (6.1a). Alternatively $-\Phi^t$ can be computed in each period as the maximization of

$$\begin{matrix} [B^t - c^t x^t] & + & [-\psi_1^t(s)] & + & \psi_2^t(d) \\ \left(\begin{array}{c}\text{measure of the}\\ \text{sectoral benefit}\\ \text{in period t}\end{array}\right) & + & \left(\begin{array}{c}\text{alternative primary}\\ \text{supplies producers'}\\ \text{surplus in period t}\end{array}\right) & + & \left(\begin{array}{c}\text{end-use goods}\\ \text{consumers' surplus}\\ \text{in period t}\end{array}\right) \end{matrix}$$

where B^t is the sectoral budget in period t, in which case

$$K^t = -B^t$$

in (6.1a). We shall omit from hereon the constant K^t in (6.1) since it does not affect the optimal solution to (6.1) and it vanishes when we define the depletable resource suppliers' *revenue function* as

$$\beta^t(r) \equiv \Phi^t(0) - \Phi^t(r) \tag{6.6}$$

It should be kept in mind however that it plays a crucial role in guaranteeing a meaningful cost function, or in other words, $\Phi^t(r) \geq 0$.

The Kuhn-Tucker optimality conditions to the market-sectoral problem (6.1) embody the conditions for the first level of equilibrium, namely an economic equilibrium between the economic sector and the markets for end-use goods and for alternative primary supplies. Necessary and sufficient conditions for optimality in (6.1) require, for each fixed $\hat{r}^t$, the existence of $\hat{x}^t, \hat{s}^t, \hat{d}^t, \hat{\pi}^t, \hat{w}^t$ and $\hat{u}^t$, t=1,2,...,T,satisfying

$$\nabla\psi_1^t(\hat{s}^t) - \hat{w}^t \geq 0 \qquad \text{with equality if } \hat{s}_i^t > 0,\ t=1,2\ldots T \qquad (6.7a)$$

$$\nabla\psi_2^t(\hat{d}^t) - \hat{u}^t \leq 0 \qquad \text{with equality if } \hat{d}_j^t > 0,\ t=1,2\ldots T \qquad (6.7b)$$

$$-\pi^t\rho^t - \hat{w}^t A_1^t + \hat{u}^t A_2^t \leq c^t \quad \text{with equality if } \hat{x}_k^t > 0,\ t=1,2\ldots T \qquad (6.7c)$$

$$\hat{\pi}^t(\rho^t\hat{x}^t - \hat{r}^t) = 0 \qquad t=1,2,\ldots,T \qquad (6.7d)$$

$$\hat{w}^t(A_1^t\hat{x}^t - \hat{s}^t) = 0 \qquad t=1,2,\ldots,T \qquad (6.7e)$$

$$\hat{u}^t(A_2^t\hat{x}^t - \hat{d}^t) = 0 \qquad t=1,2,\ldots,T \qquad (6.7f)$$

$$\rho^t\hat{x}^t \leq \hat{r}^t \qquad t=1,2,\ldots,T \qquad (6.7g)$$

$$A_1^t\hat{x}^t - \hat{s}^t \leq 0 \qquad t=1,2,\ldots,T \qquad (6.7h)$$

$$A_2^t\hat{x}^t - \hat{d}^t \geq 0 \qquad t=1,2,\ldots,T \qquad (6.7i)$$

$$\hat{s}^t \geq 0,\ \hat{d}^t \geq 0,\ \hat{u}^t \geq 0,\ \hat{x}^t \geq 0,\ \hat{w}^t \geq 0,\ \hat{\pi}^t \geq 0 \qquad t=1,2\ldots T \qquad (6.7j)$$

The equilibrium interpretation of conditions (6.7) is well known. As noted by Shapiro [79] the connection between econometric submodels and the sectoral linear programming submodel is effected by conditions (6.7a) and (6.7b). These require the equality of the shadow prices with supply and demand prices at positive levels but allows $\hat{w}_i^t \geq \hat{p}_{s_i}^t$ if $\hat{s}_i^t = 0$ and $\hat{u}_j^t \leq \hat{p}_{d_j}^t$ if $\hat{d}_j^t = 0$.

For the second level of equilibrium, between the economic sector and the supplier of the depletable resource, we require the equilibrium condi-

tions (2.9), or in other words that

$$\hat{r}^t \geq 0 \qquad t=1,2,\ldots,T \tag{6.8a}$$

$$\sum_{t=1}^{T} \hat{r}^t \leq R \tag{6.8b}$$

$$\begin{aligned} V_T(R) = \sum_{t=1}^{T} \alpha^{t-1}\{\phi^t(0) - \phi^t(\hat{r}^t) - g^t(\sum_{j=1}^{t-1} \hat{r}^j, \hat{r}^t) \\ + \alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} \hat{r}^t) \end{aligned} \tag{6.8c}$$

The simultaneous satisfaction of conditions (6.7) and (6.8) is what we define as a *general equilibrium*.

An iterative approach is proposed to solve for the general equilibrium given by conditions (6.7) and (6.8). As before, at iteration n, a piecewise linear approximation to the supplier's concave revenue function β^t, is used to solve the supplier's problem (2.1). The resulting supply schedule $(r^{1,n}, r^{2,n}, \ldots, r^{T,n})$ is passed to the market-sectoral problems (6.1) in each period, t=1,2,...,T. The market sectoral problems (6.1) for periods t=1,2,...,T are optimized at these resource levels by interacting with the econometric submodels by any applicable decomposition method as suggested by Shapiro [79]:

i) subgradient optimization (Held and Karp [39], Fisher [20], and Polyak [68]),

ii) primal-dual ascent algorithm (Fisher and Shapiro [23],

Fisher, Northrup and Shapiro [22], and Lemarechal [55]),

iii) simplicial approximation (Scarf and Hansen [74]),

iv) generalized linear programming (Lasdon [54]).

This will yield sectoral costs $\phi^t(r^{t,n})$ and shadow prices $\pi^{t,n+1}$, for $t=1,2,\ldots,T$. At this point either an equilibrium solution satisfying (6.7) and (6.8) has been found or a new linear segment is added to improve the upper-bound approximations. This is also depicted in Figure 6.1.

The convergence properties of the above iterative scheme can be seen to follow also from Theorem 2.2 and the application of Geoffrions' [26] generalized Benders' decomposition to the problem

$$\max \sum_{t=1}^{T} \alpha^{t-1}\{\phi^t(0) - c^t x^t - \psi_1^t(s^t) + \psi_2^t(d^t) - g^t(\sum_{j=1}^{t-1} r^j, r^t)\}$$

$$+ \alpha^T \beta^{T+1}(R - \sum_{t=1}^{T} r^t)$$

$$\begin{aligned}
\text{s.t.} \quad & \rho^t x^t \le r^t && t=1,2,\ldots,T \\
& A_1^t x^t - s^t \le 0 && t=1,2,\ldots,T \qquad (6.9)\\
& A_2^t x^t - d^t \ge 0 && t=1,2,\ldots,T \\
& \sum_{t=1}^{T} r^t \le R && t=1,2,\ldots,T \\
& 0 \le x^t,\ 0 \le s^t,\ 0 \le d^t \text{ and } 0 \le r^t && t=1,2,\ldots,T
\end{aligned}$$

For a fixed non-negative supply schedule $(r^1, r^2, \ldots, r^T)$ satisfying $\sum_{t=1}^{T} r^t \leq R$ problem (6.9) is separable into T subproblems, the market-sectoral problems (6.1) generating Φ^t. The master problem is the supplier's problem (2.1), with the supplier's revenue function β^t as defined by (6.6).

Problem (6.9) can be interpreted as a maximization of discounted surplus. Four different surpluses can be identified in the objective of (6.9). These are: the alternative primary suppliers' producers' surplus, the end-use goods consumers' surplus, the economic sector's surplus and the depletable resource producers' surplus.

6.III. Quadratic Market-Sectoral Problems.

In Chapter 2 we developed a convergent iterative approach to solve for equilibrium for piecewise linear as well as general convex cost functions Φ^t. While for general convex sectoral problems, this seems to be the only scheme of practical viability, for linear programming sectoral problems, parametric programming is applicable. In the latter case the argument for an iterative approach was based on the fact that a large number of linear segments combined with a long planning horizon would make parametric linear programming expensive enough to be considered not viable. However it is conceivable that a combination of parametric linear programming and the linear piece-generation scheme might be the "optimal" approach.

Apparently there is another class of problems where parametric programming can be efficiently handled. This would be the case when Φ^t is

computed from a convex quadratic programming problem, which results when the alternative primary supplies (6.2a) and the end-use demands (6.2b) econometric specifications in for the market-sectoral problem (6.1) are given by

$$p_s^t = W_1^t s^t + w_1^t$$

$$p_d^t = W_2^t d^t + w_2^t$$

respectively. We assume that W_1 is symmetric positive semi-definite and that W_2 is symmetric negative semi-definite matrices. This case was partially considered by Kennedy [50].

Dropping the time superscripts for simplicity, the market-sectoral problem (6.1) would be then given by

$$\begin{aligned} \Phi(r) = \min\ & c'x + \frac{1}{2}s'W_1s + w_1's - \frac{1}{2}d'W_2d - \frac{1}{2}w_2'd \\ \text{s.t.}\ & \rho x \le r \\ & A_1x - s \le 0 \\ & A_2x - d \ge 0 \\ & 0 \le x\ ,\ 0 \le s \text{ and } 0 \le d \end{aligned} \tag{6.10}$$

where the prime (') denotes the transpose. Equivalently, in terms of its Dorn's [17] dual (see Stoer and Witzgall [86]),

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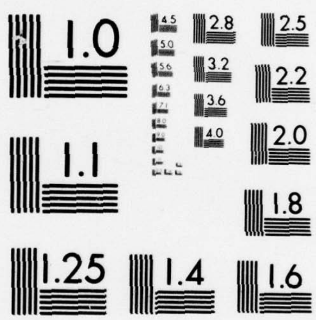

MICROCOPY RESOLUTION TEST CHART
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$$\Phi(r) = \max \quad \pi r - \frac{1}{2}s'W_1 s + \frac{1}{2}d'W_2 d$$

$$\text{s.t.} \; - \pi\rho - vA_1 + uA_2 \leq c$$

$$v - W_1 s \leq w_1$$

$$u - W_2 d \geq w_2$$

$$0 \leq \pi,\; 0 \leq u,\; 0 \leq v,\; 0 \leq s \text{ and } 0 \leq d$$

The shape of the cost function Φ^t, generated by the market-sectoral problem (6.10), is piecewise quadratic. This will lead to demand functions (see section 3.III) that are piecewise linear as depicted below.

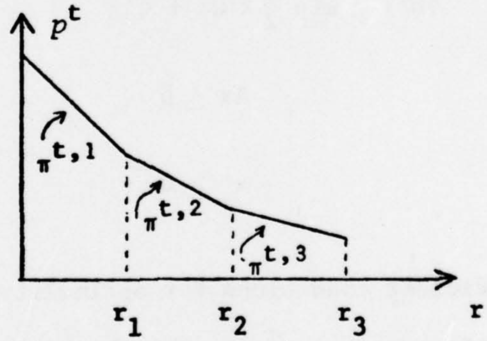

Figure 6.2

This is in contrast with the staircase demand functions that result from the LP-sectoral problem (2.10) as shown in Figure 6.2.

In order to rewrite (6.10) in the standard form for quadratic

programming problems, we let

$$W = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & -W_2 \end{bmatrix} \qquad A = \begin{bmatrix} \rho & 0 & 0 \\ A_1 & -I & 0 \\ -A_2 & 0 & I \end{bmatrix}$$

$$c = \begin{bmatrix} c \\ w_1 \\ -w_2 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ s \\ d \end{bmatrix} \qquad b = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$

where I denotes the identity matrix and W is symmetric and positive semi-definite. Hence generating Φ^t in (6.10) can be seen as equivalent to determining $\Phi(b)$ for $b \geq 0$ in the quadratic program

$$\begin{aligned} \Phi(b) &= \min \tfrac{1}{2} x'Wx + c'x \\ Ax &\leq b \\ x &\geq 0 \end{aligned} \tag{6.11}$$

Necessary and sufficient conditions for optimality of x in (6.11) require the existence of vectors y and u, v such that

$$\begin{cases} u = Wx - A'y + c \\ v = Ax - b \\ 0 \leq u,\ 0 \leq x,\ 0 \leq y,\ 0 \leq v \\ x'u = 0,\ y'v = 0 \end{cases} \tag{6.12}$$

The redefinitions

$$W = \begin{pmatrix} u \\ v \end{pmatrix}, \quad q = \begin{pmatrix} c \\ -b \end{pmatrix}, \quad M = \begin{pmatrix} W & -A' \\ A & 0 \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$

establish (6.12) as a linear complementary problem

$$\begin{cases} w = q + Mz & w \geq 0, \quad z \geq 0 & (6.13a) \\ z\,w = 0 & & (6.13b) \end{cases}$$

which can be solved by complementary pivoting as proposed by Cottle and Dantzig [8].

The algorithm for positive-definite matrix M consists of a sequence of *major cycles* initiated with a complementary basic solution

$$(w;z) = (q;0)$$

If $q \geq 0$, the procedure is terminated. Otherwise for any $w_e = q_e < 0$, the complement z_e of the selected negative basic variable is increased until it is either blocked by a positive basic variable decreasing to zero or by the negative w_e increasing to zero. At this point the blocking variable is pivoted out of the basis by pivoting in its complement and the major cycle continues. The major cycle is only terminated when w_e drops out of the basis. Further details are omitted.

A slight modification of the above procedure was suggested later by Cottle and Dantzig [7] to deal with the positive semi-definiteness of M.

While the latter would be the required procedure for the quadratic sectoral-market problem (6.10), we feel that the parametric programming would have to be modified slightly in the same direction. Since we shall not enter into pivoting considerations, without loss of generality, we concentrate here on the positive definite case.

Partition $[I,-M]$ into $[B,N]$ where B is a square matrix formed by the columns of $[I,-M]$ corresponding to the basic variables and N by those columns corresponding to the complementary non-basic variables. Relabeling the variables if necessary, once the process is terminated the system (6.13a) is rewritten in canonical form as

$$\bar{w} = \bar{q} + \bar{M}\,\bar{z}$$

with $\bar{q} = B^{-1} q$ and $\bar{M} = -B^{-1} N$. The solution $(\bar{w};\bar{z}) = (\bar{q};0)$ has $\bar{q} \geq 0$. Since it is a feasible complementary basic solution of (6.13), it is optimal for (6.11).

Consider now a change in the left-hand side of (6.13) from q to $q + \theta\delta$, where δ is a parametrization vector and $\theta \geq 0$. Defining $\nu = B^{-1}\delta$, and letting

$$B^{-1}(q + \theta\delta) = \bar{q} + \theta\nu$$

we observe that if $\nu_i \geq 0$ for all i then the complementary basic solution $(\bar{w};\bar{z}) = (\bar{q} + \theta\nu;0)$ is feasible and hence optimal for (6.11) with modified parameters $q + \theta\delta$, for all $\theta \geq 0$.

Conversely, if $\nu_i < 0$ for some i, then as θ is increased one or more basic variables may become negative. The maximum increase in θ that does not affect feasibility is

$$\theta_1 = \min_{\nu_i < 0} \left\{ \frac{\bar{q}_i}{-\nu_i} \right\} = \frac{\bar{q}_e}{-\nu_e}$$

For $\theta > \theta_1$, the complementary basic solution $(\bar{w};\bar{z}) = (\bar{q} + \theta\nu;0)$ has $w_e = \bar{q}_e + \theta\nu_e < 0$. A major cycle of the complementary pivoting procedure can be initiated at this point, to regain feasibility.

<u>Theorem 6.1</u>: Barring degeneracy, feasibility is regained after a finite number of complementary pivots.

Proof: This is so because the maximum number of variables that may become negative as q changes to $q + \theta\delta$ is finite. The number of basic variables with negative values at the end of a major cycle is at least one less than at the beginning of the major cycle. Each major cycle terminates after a finite number of complementary pivots since the number of bases formed from $[I,-M]$ is finite. $||$

A numerical example should clarify the parametrization procedure.

<u>**Example 6.1**</u>: Consider determining for $r \geq 0$.

$$\phi(r) = \min \; [-3 \; -4 \; -1] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \frac{1}{2} [x_1 \; x_2 \; x_3] \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{s.t.} \quad -x_1 - 2x_2 - x_3 \geq -r$$

$$x_1, x_2, x_3 \geq 0.$$

Starting with $r = 4$ and letting $z_i = x_i$, $i=1, 2, 3$, $z_4 = y_1$, $w_i = u_i$, $i=1,2,3$ and $w_4 = v_1$, the initial complementary basic solution is given by

$$w_1 = -3 + z_1 + 2z_2 - z_3 + z_4$$

$$w_2 = -4 + 2z_1 + 4z_2 - 2z_3 + 2z_4$$

$$w_3 = -1 - z_1 - 2z_2 + z_3 + z_4$$

$$w_4 = 4 - z_1 - 2z_2 - z_3$$

The complementary pivoting procedure terminates with $x_1 = 5/2$, $x_2 = 0$ and $x_3 = 3/2$, and

$$w_2 = 2 + 2w_1$$

$$z_4 = 2 + 1/2w_1 + 1/2w_3$$

$$z_3 = 3/2 - 1/4w_1 + 1/4w_3 - 1/2w_4$$

$$z_1 = 5/2 + 1/4w_1 - 1/4w_3 - 1/2w_4 - 2z_2$$

The complementary basis is given by

$$B = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & -2 & 2 & -2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \text{with } B^{-1} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 \\ 1/4 & 0 & -1/4 & 1/2 \\ -1/4 & 0 & 1/4 & 1/2 \end{pmatrix}$$

Let now $r = 4 + \theta$, $\theta \geq 0$, this corresponds to

$$\delta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ +1 \end{pmatrix} \qquad \text{and} \qquad \nu = B^{-1}\delta = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

Since $\nu_i \geq 0$, i=1, 2, 3, 4, for $\theta \geq 0$, the complementary basic solution

$$w_2 = 2 + \theta\nu_1 = 2$$

$$z_4 = 2 + \theta\nu_2 = 2$$

$$z_3 = 3/2 + \theta\nu_3 = 3/2 + 1/2\theta$$

$$z_1 = 5/2 + \theta\nu_4 = 5/2 + 1/2\theta$$

and $w_1 = w_3 = w_4 = z_2 = 0$ remains feasible for (6.13) and hence optimal for the original problem. In terms of the original variables, the solution is $x_1 = (r+1)/2, x_2 = 0$ and $x_3 = (r-1)/2$ for $r \geq 4$.

For $r = 4 - \theta,\ \theta \geq 0$,

$$\delta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \text{yields} \quad \nu = B^{-1}\delta = \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix}$$

and since $\min\left\{\frac{3/2}{1/2}, \frac{5/2}{1/2}\right\} = 3$, for $0 \leq \theta \leq 3$, or equivalently $1 \leq r \leq 4$, the complementary basic solution

$$\begin{aligned} w_2 &= 2 \\ z_4 &= 2 \\ z_3 &= 3/2 - 1/2\theta \\ z_1 &= 5/2 - 1/2\theta \end{aligned}$$

remains feasible. As $3 < \theta \leq 5$, the basic variable z_3 becomes negative. A major cycle is initiated by increasing w_3, yielding

$$\begin{aligned} w_2 &= 2 + 2w_1 \\ z_4 &= \theta - 1 + w_1 + 2z_3 + w_4 \\ w_3 &= 2\theta - 6 + w_1 + 4z_3 + 2w_4 \\ z_1 &= 4 - \theta - z_3 - w_4 - 2z_2 \end{aligned}$$

For $3 < \theta \leq 4$, the complementary solution

$$(w_2, z_4, w_3, z_1; w_1, w_4, z_2, z_3) = (2, \theta - 1, 2\theta - 6, 4 - \theta; 0,0,0,0)$$

is feasible and hence optimal. In terms of the original variables

$x_1 = r, x_2 = x_3 = 0$ is the optimal solution for $0 \le r \le 1$.

This gives

$$\phi(r) = \begin{cases} \dfrac{r^2}{2} - 3r & 0 \le r \le 1 \\ -2r - 1/2 & 1 \le r \end{cases} .$$

While parametric programming was here illustrated for the complementary pivoting procedure, apparently it can be easily adapted to any other procedure directly or indirectly based on the linear complementarity conditions (6.13). It is felt that in Wolfe's [94] simplex method for quadratic programming, parametric programming can be performed by a modified dual simplex algorithm.

6.IV. Non-Separable Sectoral Problems

The development of the sectoral-supplier equilibrium in the previous chapters assumed that the cost of meeting demand (2.6), ϕ^t, in each period is solely determined by the amount of depletable resource available in that period. The separability over time of the sectoral problems has led to revenue functions (2.8) whose cross time effects are zero, or in other words,

$$\frac{\partial \beta^i}{\partial r^j} = 0 \qquad \text{for } i \neq j$$

The rationale for this situation is the sectoral intertemporal decisions (e.g., capacity expansion) being taken either prior or posterior to agreeing with the supplier on a depletable resource supply schedule. In the first case the sectoral problems would be provided with the optimal capacity addition schedule as exogenous data. In the second case, the sectoral problems would not take into account the capacity expansion process and these additional cost reductions would not be reflected in the supplier's revenue functions. It is when the decisions on the sectoral capacity expansion and the depletable resource supply schedule are made simultaneously that difficulties, due to the non-separability of the cost function, will arise. For illustration let

$\Phi(r^1, r^2, \ldots, r^T)$ = present value of the cost of meeting demand from period 1 through T when r^t units of the depletable resource is available to the sector in each period $t=1,2,\ldots,T$

be computed by the *multiperiod linear programming sectoral problem*

$$\Phi(r^1, r^2, \ldots, r^T) = \min \sum_{t=1}^{T} \alpha^{t-1} \{c^t x^t + f^t s^t + q^t y^t\} \tag{6.14a}$$

$$\rho^t x^t \leq r^t \qquad t=1,2,\ldots,T \tag{6.14b}$$

$$A_1^t x^t - s^t \leq 0 \qquad t=1,2,\ldots,T \tag{6.14c}$$

$$A_2^t x^t \geq d^t \qquad t=1,2,\ldots,T \tag{6.14d}$$

$$0 \le s^t \le \bar{s}^t \qquad t=1,2,\ldots,T \qquad (6.14e)$$

$$0 \le x^t \le y^o + \sum_{j=1}^{t-1} y^j \qquad t=1,2,\ldots,T \qquad (6.14f)$$

where y^t denotes the additional capacity installed in period t purchased at a unit cost q^t, and y^o the initial capacity. This reformulation resembles the transition from BESOM [42] to DESOM [62] (see section 1.II.3). The mere existence of constraints (6.14f) will in general impair the separability of the sectoral problem (6.14). It is not clear, in the present case, how the present value of the sectoral cost-savings,

$$\Phi(0,0,\ldots\ldots,0) - \Phi(r^1,r^2,\ldots\ldots,r^T)$$

could be disaggregated to yield the individual contributions of each period supply. In the spirit of this thesis, one possibility to be further explored among several alternatives available to define in each period the supplier's revenue function would be

$$\beta^t(r^1,r^2,\ldots\ldots,r^{t-1},r) = \alpha^{1-t}[\,\Phi(r^1,r^2,\ldots,r^{t-1},0) - \Phi(r^1,r^2,\ldots,r^{t-1},r)] \qquad (6.15)$$

In the case of (6.15), the present value of the supplier's revenue would equal the present value of the sectoral cost-savings but, due to the dependence of β^t on $(r^1,r^2,\ldots,r^{t-1})$, the advantageous dynamic programming reformulation of the supplier's problem is not directly applicable.

Apparently the dependence of the sectoral-cost savings on variables other than the supply in that period is uniquely a function of the assump-

tions imposed on the sectoral model. Even though it is not obvious which formulation of the sectoral problem might lead to a cost of meeting demand in each period that would depend only on the maximum amount of the depletable resource available until the end of the planning horizon, S, and the present period supply, r, this would be the case where the definition

$$\beta^t(S,r) = \Phi^t(S,0) - \Phi^t(S,r)$$

would not modify drastically our analysis.

Still another possibility would be the formulation of the sectoral problem also as a dynamic programming problem in the state space (Y,S) that could encompass among other things capacity stocks and inventories. However how these two dynamic programming recursions (the sectoral and the depletable resource supplier) would interact in the state space (Y,S) is not obvious.

6.V. Smoothing the Cost of Meeting Demand

In the derivation of the cost of meeting demand at alternative depletable resource levels r, Φ^t was ideally computed by solving a sectoral problem parametrically in r. The sectoral problems discussed in section 2.III will lead to convex functions Φ^t, but these will not be differentiable in general. The linear programming formulation (2.10) would lead to a piecewise linear function and the general convex formulation (2.16) may lead to piecewise non-linear functions. While the non-differentiability of Φ^t does not affect algorithmically the solution procedures to the sectoral-supplier equilibrium, it has several drawbacks

from an economist standpoint. The non-existence of derivatives everywhere will not permit a clear evaluation of the effect of changes in the alternative primary supplies prices and of substitution effects by means of price-elasticities and substitution elasticities.

Griffin [29] proposed a procedure to dissassociate the engineering technical details imbedded, in our case, in the sectoral problems, leading to non-differentiable engineering technologies, from the smoother long-run model. The "pseudo-data," generated by a process model or a sectoral model would then be utilized to estimate a statistical cost function, offering a differentiable approximation to the engineering technology cost function.

As mentioned in section 2.II, the function Φ^t can be seen with respect to the depletable resource, as a short-run cost function. At a fixed level of resource availability, r , Φ gives the minimum cost configuration between alternative primary inputs, s, and non-primary inputs, x. This was, indeed, the line of reasoning in the formulation of the convex sectoral problem (2.16). This analogy will permit that closed-form solutions be obtained for Φ in special cases.

Let

$$\begin{aligned} \Phi(r) = \min\ & c\,x + f\,s \\ & F(d;\ \{x,s,r\}) \le 0 \\ & x \ge 0,\quad s \ge 0 \end{aligned}$$

where F is given by a separable single-valued convex multiple-output production function. For its properties see Hasenkamp [37].

We shall treat now a special case. When F is separable into

$$F(d;\ \{x,s,r\}) = h_o(d) - h_i(\{x,s,r\})$$

where h_o and h_i are respectively convex output and concave input functions and h_i is a constant elasticity of substitution input function, given by

$$h_i(\{x,s,r\}) = \left(\sum_i \nu_i x_i^{\frac{\sigma}{\sigma-1}} + \sum_j \nu_j s_j^{\frac{\sigma}{\sigma-1}} + \nu_{N+1}\, r^{\frac{\sigma}{\sigma-1}}\right)^{\xi(\frac{\sigma-1}{\sigma})}$$

where all $\nu > 0$, ξ denotes the returns to scale parameter and σ denotes the elasticity of substitution between any two inputs for all input combinations, the short-run cost-function is given by Hasenkamp [37] as

$$\Phi(r) = Y\left\{h_o(d)^{\frac{\sigma}{(\sigma-1)\xi}} - \nu_{N+1} r^{\frac{\sigma}{\sigma-1}}\right\}^{(\frac{\sigma-1}{\sigma})}$$

where $Y = \left\{\sum_i (\nu_i)^{1-\sigma}(c_i)^{1/\sigma} + \sum_j (\nu_j)^{1-\sigma}(f_j)^{1/\sigma}\right\}$.

It is easy to verify that an input may only be inessential if $\sigma > 1$, ruling out the Cobb-Douglas case. Under this assumption, the cost-savings function is given by

$$\Phi(0) - \Phi(r) = Y\left[\left[h_o(d)\right]^{1/\xi} - \left\{\left[h_o(d)\right]^{\frac{\sigma}{(\sigma-1)\xi}} - \nu_{N+1} r^{\frac{\sigma}{\sigma-1}}\right\}^{(\frac{\sigma-1}{\sigma})}\right]$$

This example suggests that the same method employed by Griffin [29]

to smooth a non-differentiable cost function might be applicable to smooth the sectoral cost of meeting demand function, Φ. However, it is not obvious how the flexible translog cost function developed by Christensen, Jorgenson and Lau [5], which purports to be a second-order local approximation to any technology would be extended to a short-run translog cost functional specification. This difficulty stems from the fact that the translog production, profit and cost functions, are not derived from each other by duality considerations. Instead, they have been proposed to keep the same analytic form.

The advantages from our standpoint, could mean the elimination of running simultaneously one sectoral problem for each time period in the planning horizon. Instead, a single functional form would give the cost of meeting demand at alternative price levels for the other inputs, efficiencies in resource utilization, and levels of end-use demand. The scheme would then be modified as depicted in Figure 6.3.

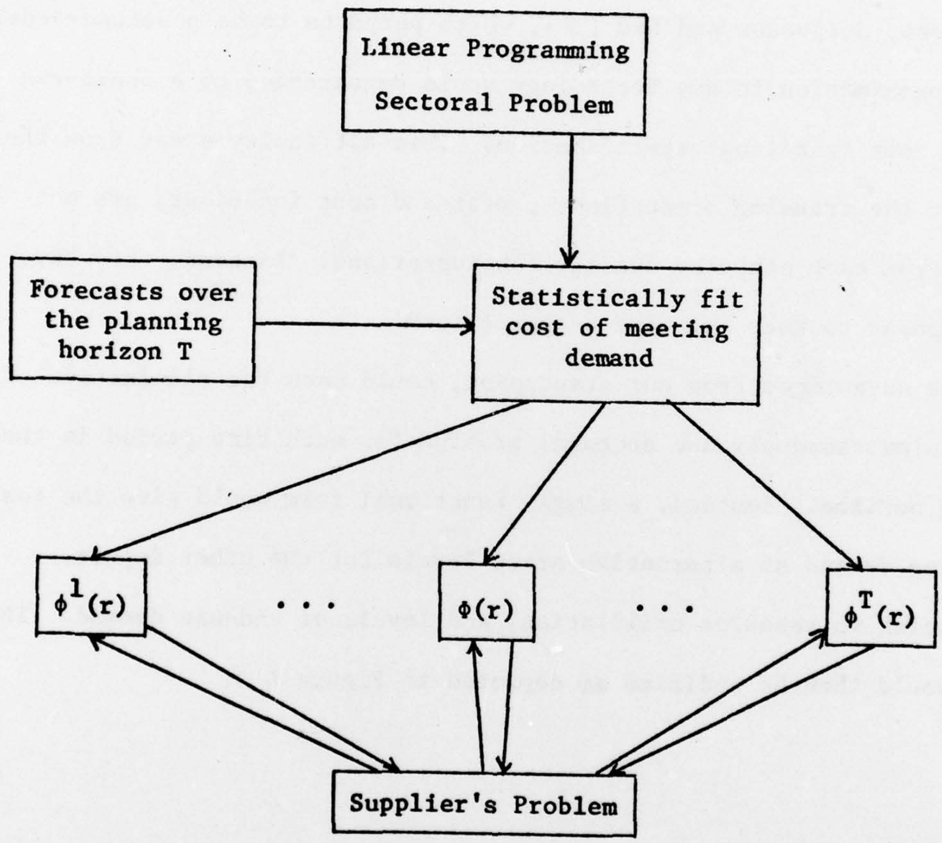
Linear Programming
Sectoral Problem
Forecasts over the planning horizon T
Statistically fit cost of meeting demand
$\phi^1(r)$
. . .
$\phi(r)$
. . .
$\phi^T(r)$
Supplier's Problem

Figure 6.3

Chapter 7: Conclusions

The emergence of the OPEC cartel, culminating in the Arab oil embargo of 1973, generated a world-wide urge for the development and implementation of detailed energy technology and policy assessment models. Existing models together with the newly designed models covered a wide spectrum of energy-related issues. A diversity of techniques has been utilized in the development of the different models. An evident following step would be the definition of a unified framework that would combine features of this variety of models. Model integration is apparently the future trend in energy modeling. This will provide at each step a further detailed macro-vision of the structure of the energy sector.

In this thesis we set a framework for the integration of sectoral (demand) and supply models. Our particular emphasis has been placed on exhaustible resources, in which category we find most of the energy sector primary resources. Compared to early sectoral models, those developed here are dynamic as we are mainly concerned with the allocation over time of non-augmentable resource stocks. Opposed to empirical or econometric models (e.g. Hudson and Jorgenson [46]) our approach is inherently normative. Supply and demand models are formulated as optimization problems. Mathematical programming decomposition concepts permit the direct and separate treatment of the behavioral aspects of all market participants. This is in contrast with current large-scale model integration (e.g. Hoffman and Jorgenson [43]) where normative behavior is indirectly treated, establishing the underlying basis for aggregate econometric models. We

can observe along this thesis numerous advantages of disclosing the normative bases of the models in accomplishing the integration of depletable resource supply and demand models. Of course, this can only be accomplished at the expense of a well-behaved aggregate description of the economic activity and/or of consumer and producer response. The normative approach also reveals some of the major concerns in the economic theory of exhaustible resources; namely, optimal paths to final exhaustion, finite versus infinite time depletion, the effect of depletion on price trends, scarcity rents or user's costs, to cite a few.

Our conceptual framework decomposes the problem of allocation of fixed resource stocks into supply and sectoral (demand) subproblems, and links them by means of equilibrium conditions. The evaluation of the intertemporal biases in extraction rates that result from diverse supply and demand market structures can be accomplished by the consideration of more elaborate forms of sectoral and suppliers problems. To a certain extent the efficient resource allocation can be compared with the allocations reflecting the different degrees of monopsony and/or monopoly powers of the buying and selling markets. Also, our methodology seems particularly suitable to the direct assessment of the impact of technological change. The detailed engineering process sectoral models play an active role in this scheme. Long-run changes from technologies considered for implementation can be easily incorporated into this scheme because of the flexibility permitted in the design of the time independent sequence of sectoral models. Energy programs and regulatory economic policies to stimulate energy conservation, to inhibit or expedite primary resource

depletion and to sponsor research and development (e.g. price controls, tax depletion allowances, pollution charges) can be analyzed. This requires only the recognition of the affected economic agent and the regulatory effect on the quantities and/or shadow prices iteratively exchanged between the economic sectors and the depletable resource suppliers.

Highly aggregated models and empirical models in general do not allow the direct consideration of different modes of behavior toward the uncertainties faced by the sectors in the economy (e.g. demand for end-use energy products, timing of technological change) and the resource owners (e.g. the amount of reserves and rates of extraction). This is so because their normative aspects are not carried explicitly. In this thesis, instead, we are able to model different attitudes toward uncertainty by prescribing different behaviors to the affected economic agents.

Currently available intertemporal energy models are not representative of an effort to consider infinite planning horizons. However, it is widely recognized that the impact of present resource extraction decisions on the welfare of all future generations cannot be neglected. Based on the premise that the time discounting will efface approximation errors, the arbitrary truncation of the infinite planning horizon is usual practice. In our models, the advantageous dynamic programming formulation of the supply model will permit us to exactly accommodate the infinite planning horizon. This is of particular interest in the coupling of transient and stationary economy stages.

The modeling methodology considered in this work is justified on the basis of economic theory in addition to being operational, in the sense

that we address the technical details of the integration of supply and sectoral models. Following the presentation by Modiano and Shapiro [65] which established the ground rules of this thesis and illustrated more sophisticated mathematical programming techniques in intertemporal energy modeling, more recent work (e.g. Pariente [67], Shapiro and White [80]) suggests that this is a powerful area for further research.

The normative models of depletable resources that we propose here should be seen as an attempt to bridge the existing gap between energy modeling and the economic theory of exhaustible resources. We recognize, however, that the joint effort and involvement of several researchers will be required to bridge this gap completely and build an operationally unified and integrated economic framework for energy modeling.

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